OPTIMUM INDUSTRIAL MANAGEMENT SYSTEMS
BY THE MAXIMUM PRINCIPLE

By

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1. GENERAL INTRODUCTION

A remarkable growth of interest in problems of dynamic optimization has given rise during the past decade to a number of methods useful for rendering systems optimal. One such method is Pontryagin's maximum principle.

Originally formulated in 1956 by the Russian mathematician and his associates (10), the maximum principle was intended for the optimization of continuous control systems. In 1959, the first attempt to extend the maximum principle to the optimization of stagewise processes was made by Rozonoer (11). Subsequent versions of the discrete maximum principle were then advanced by Chang (1), Katz (9), and by Fan and Wang (2).

The application of the maximum principle to management and operations research is still very limited. Transportation problems (3,5), a capital investment problem (allocation of a resource) (6), and a one-dimensional production problem (7) are examples of the discrete cases which have been recently investigated.

The main objective in this thesis is to demonstrate the applicability of the maximum principle to other problems in the area of industrial engineering and management, concentrating the attention mainly on those problems belonging to the continuous case. It is not the primary intention, therefore, to introduce new mathematical models. Rather, it is intended to apply the maximum principle in order to optimize already developed models, and functional variations of these models. When appropriate or

necessary, numerical examples are also presented for further clarification of the treatment.

The basic algorithms of the discrete and the continuous maximum principle are presented first, and then the discrete version is applied in order to optimize the temperatures of a multistage heat exchanger train (2). The treatment of this system leads to a two-point boundary value problem whose solution is demonstrated in detail.

A model for sales response to advertising developed by Vidale and Wolfe (12) is then treated by the continuous maximum principle. The optimum solution of this system leads to three key advertising policies. The linear constraint on the response function is then removed and the maximum principle is again applied to the modified model.

Next, a continuous model for production planning presented by Holt _et_ _al_. (4) and to which the maximum principle was applied by Hwang and Fan (8) is studied. Finally, two models for the optimization of equipment investment based on the net present value are treated by the maximum principle. Two numerical examples are included in the analysis of these two models.

The efficiency of the maximum principle in dealing with this class of problems is not compared with that of other methods. The reason for this is that the application of the maximum principle to this sort of problems has not left the incipient stages of development. This is a new technique and as such, due refinements and further developments must take place before any comparisons of computational efficiency can be made.

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2. THE ALGORITHMS OF THE MAXIMUM PRINCIPLE

INTRODUCTION

The two basic algorithms of the maximum principle are presented in this section. Although the applicability of these algorithms may appear to be limited, both algorithms can be extended to handle a variety of problems encountered in practice, for example, processes with fixed end points, processes with choice of initial values, processes with choice of extra parameters, processes with arbitrary final measures as the objective function and so on. The details of these extensions are given in Ref. 1 for the discrete maximum principle and in Ref. 2 for the continuous maximum principle. In these references the algorithms for the optimization of complex systems are also treated.

THE DISCRETE MAXIMUM PRINCIPLE (1, 3)

A multistage decision process may be considered as an abstract notion by which a large number of human activities can be presented. Since a multistage decision process is an entity consisting of a finite number of stages, the nature of the process is completely determined by the types of stages from which the process is formed and by the way the stages are interconnected.

A schematic representation of a simple multistage process is shown in figure 1. The process consists of N stages connected in series. A stage may represent any real or abstract entity (for example, a space unit, a time period, or an economic

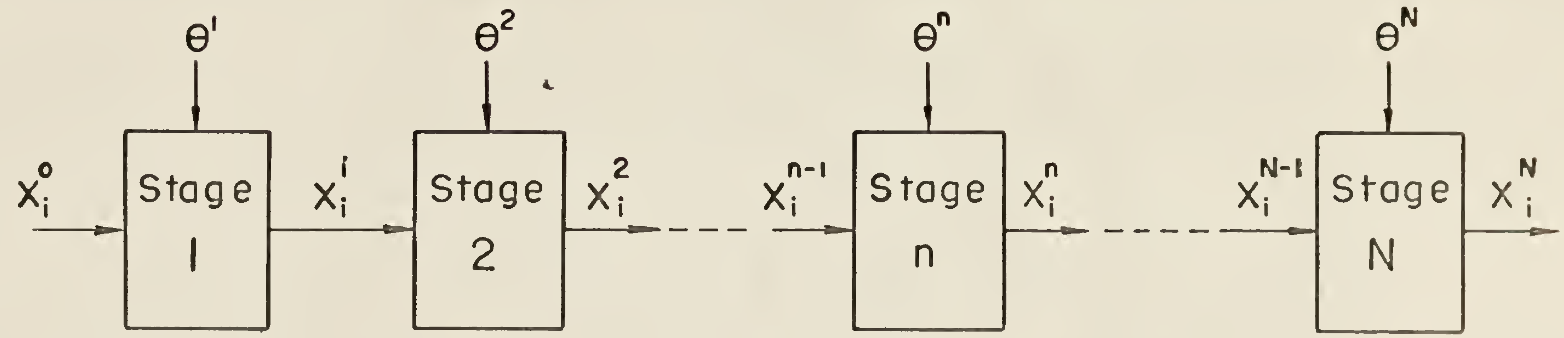

Fig. 1. Multistage decision process

activity) in which a certain transformation takes place. The state of the process stream denoted by the s-dimensional vector, $x = (x_1, x_2, \ldots, x_s)$ is transformed at each stage according to an r-dimensional decision vector, $\theta = (\theta_1, \theta_2, \ldots, \theta_r)$, which represents the decisions made at that stage. The transformation of the process stream at the n-stage is described by a set of performance equations,

$$x_i^n = T_i^n(x_1^{n-1}, x_2^{n-1}, \ldots, x_s^{n-1}; \theta_1^n, \theta_2^n, \ldots, \theta_r^n),$$

$$x_i^0 = \alpha_i,$$

$$i = 1, 2, \ldots, s; \qquad n = 1, 2, \ldots, N,$$

or in vector form

$$x^n = T^n(x^{n-1}; \theta^n), \qquad n = 1, 2, \ldots, N, \tag{1}*$$

$$x^0 = \alpha.$$

A typical optimization problem associated with such a process is to find a sequence of θ^n, $n = 1, 2, \ldots, N$, subject to the constraints

$$\Psi_i^n(\theta_1^n, \theta_2^n, \ldots, \theta_r^n) \leq 0, \qquad \begin{array}{l} n = 1, 2, \ldots, N, \\ i = 1, 2, \ldots, r, \end{array} \tag{2}$$

which makes a function of the state variable of the final stage

$$S = \sum_{i=1}^{s} c_i x_i^N, \qquad c_i = \text{constant}, \tag{3}$$

an extremum when the initial condition $x^0 = \alpha$ is given. The function, S, which is to be maximized (or minimized) is the

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* The superscript n indicates the stage number. The exponents are written with parentheses or brackets such as $(x^n)^2$ or $[T^n(x^{n-1}; \theta^n)]^2$.

objective function of the process.

The procedure for solving such an optimization problem by the discrete maximum principle is to introduce an s-dimensional adjoint vector, z^n, and a Hamiltonian function, H^n, which satisfy the following relations:

$$H^n = \sum_{i=1}^{s} z_i^n T_i^n(x^{n-1};\ \theta^n), \qquad n = 1, 2, \ldots\ldots, N, \tag{4}$$

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}}, \quad i = 1, 2, \ldots, s;\ n = 1, 2, \ldots, N, \tag{5}$$

and

$$z_i^N = c_i, \qquad i = 1, 2, \ldots, s. \tag{6}$$

If the optimal decision vector function, $\bar{\theta}^n$, which makes the objective function S an extremum (maximum or minimum), is interior to the set of admissible decisions, θ^n,[the set given by equation (2)], a necessary condition for S to be a (local) extremum with respect to θ^n is

$$\frac{\partial H^n}{\partial \theta^n} = 0, \qquad n = 1, 2, \ldots, N. \tag{7}$$

If $\bar{\theta}^n$ is at a boundary of the set, it can be determined from the condition that H^n is (locally) extremum.

THE CONTINUOUS MAXIMUM PRINCIPLE (2, 4)

The simple continuous form of the maximum principle is concerned in general with solving problems of the following type:

Suppose that the performance equations of a control

process have the form

$$\frac{dx_i}{dt} = f_i[x_i(t),\, x_2(t), \ldots,\, x_s(t);\; \theta_1(t), \ldots,\, \theta_r(t)],$$

$$t_o \leq t \leq T,$$

$$x_i(t_o) = \alpha_i,$$

$$i = 1,\, 2, \ldots\ldots,\, s,$$

or the vector form

$$\frac{dx}{dt} = f[x(t);\; \theta(t)],\quad x(t_o) = \alpha, \tag{13}$$

where $x(t)$ is an s-dimensional vector function representing the state of the process at time t and $\theta(t)$ is an r-dimensional vector function representing the decision at time t.

The optimization problem most commonly associated with such a process is to find a piecewise continuous decision vector function, $\theta(t)$, subject to the constraints

$$\Psi_i[\theta_1(t),\, \theta_2(t), \ldots,\, \theta_r(t)] \leq 0, \quad i = 1,\, 2, \ldots,\, m, \tag{14}$$

which makes a function of the final value of the state

$$S = \sum_{i=1}^{s} c_i x_i(T), \qquad c_i = \text{constant}, \tag{15}$$

an extremum when the initial condition $x(t_o) = \alpha$ is given. The function, S, which is to be maximized (or minimized), identifies the objective function of the process.

The procedure for solving the problem is to obtain the optimum control, $\overline{\theta}(t)$, and the corresponding trajectory, $x(t)$, $t_o \leq t \leq T$, is to introduce an s-dimensional adjoint vector, $z(t)$, and a Hamiltonian function, H, which satisfy the following

relations:

$$H[z(t),\ x(t),\ \theta(t)] = \sum_{i=1}^{s} z_i f_i[x(t);\ \theta(t)], \tag{16}$$

$$\frac{dz_i}{dt} = -\frac{\partial H}{\partial x_i} = -\sum_{j=1}^{s} z_j \frac{\partial f_j}{\partial x_i},\ i = 1, 2, \ldots, s, \tag{17}$$

$$z_i(T) = c_i,\ i = 1, 2, \ldots, s. \tag{18}$$

The optimal decision vector function, $\bar{\theta}(t)$, which makes S an extremum (maximum or minimum), is the decision vector function, $\theta(t)$, which renders the Hamiltonian function, H, an extremum for almost every t, $t_o \leq t \leq T$. If the optimal decision vector function, $\bar{\theta}(t)$, is interior to the set of admissible decisions, $\theta(t)$, [the set given by equation (14)], a necessary condition for S to be an extremum with respect to $\theta(t)$ is

$$\frac{\partial H}{\partial \theta} = 0. \tag{19}$$

If $\theta(t)$ is constrained, the optimal decision vector function, $\bar{\theta}(t)$, is determined either by solving equation (19) for $\theta(t)$ or by searching the boundary of the set.

Once the decision vector function, $\theta(t)$, is chosen, the adjoint vector function, z(t), is uniquely determined by equations (17) and (18) and the initial condition at $t = t_o$, $x(t_o) = \alpha$.

We shall now present a theorem which finds application in some of the subsequent chapters. The proof of this theorem is presented in Reference 2.

<u>Theorem</u> Let $\theta(t)$, $t_o \leq t \leq T$ be a piecewise continuous

vector function satisfying the constraints given in equation (14). In order that the scalar function, S, given by equation (15) may be a maximum (or minimum) for a process described by equation (13), with the initial condition at $t = t_o$, $x(t_o) = \alpha$ given, it is necessary that there exist a nonzero continuous vector function, z(t), satisfying equations (17) and (18) and that the vector function, $\theta(t)$, be so chosen that $H[z(t),x(t),\theta(t)]$ is a maximum (or minimum) for every t, $t_o \leq t \leq T$. Furthermore, the attained maximum (or minimum) value of the Hamiltonian function, H, is a constant for every t.

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3. ITERATIVE PROCEDURE FOR THE OPTIMIZATION OF A MULTI-STAGE HEAT EXCHANGER

INTRODUCTION

The application of the maximum principle to some optimization problems often leads to the two-point boundary-value problem. The optimum design technique for a simple heat exchanger train and a refrigeration system has been described in references (1, 2) by making use of the discrete maximum principle. However, the treatment of the two-point boundary value problem has not been clearly explained.

Here, therefore, we demonstrate in detail the application of the regular Falsi method in obtaining a solution for the two-point boundary value problem when it applies to the optimal design of simple heat exchangers.

ONE DIMENSIONAL PROCESSES (1)

A multistage decision process is called a one-dimensional multistage decision process if it can be completely characterized for the purpose of optimization by a single state variable with the performance equation of the form

$$x_1^n = T(x_1^{n-1};\ \theta^n)\ , \qquad n = 1, 2, \ldots, N, \tag{1}$$

$$x_1^0 = a \tag{1a}$$

where x_1^n is the only state variable, T the transformation operator, and θ^n a r-dimensional decision vector.

In general, the objective function to be maximized is a sum of a certain function of x_1^n and θ^n over all stages of the

system such as

$$\sum_{n=1}^{N} G(x_1^{n-1};\ \theta^n).$$

The optimization problem associated with such a process is that of finding a sequence of decision variables, θ^n, n = 1, 2,..., N so as to maximize $\sum_{n=1}^{N} G(x_1^{n-1};\ \theta^n)$ with x_1^0 given. This type of problem may be treated by introducing a new state variable, x_2^n, satisfying

$$x_2^n = x_2^{n-1} + G(x_1^{n-1};\ \theta^n),\quad x_2^0 = 0,\ n = 1, 2,\ldots, N. \tag{2}$$

It can be shown that

$$x_2^N = \sum_{n=1}^{N} G(x_1^{n-1};\ \theta^n).$$

Thus, the problem becomes that of choosing a sequence of θ^n, n = 1, 2,..., N such that it maximizes x_2^N for a process described by equations (1) and (2).

To obtain the solution, a general recurrence relation for the optimal state and decision of the one-dimensional non-linear process in x_1^n will be derived from the application of the discrete maximum principle (1).

The objective function to be maximized is defined as

$$S = \sum_{i=1}^{2} c_i x_i^N = c_1 x_1^N + c_2 x_2^N = x_2^N. \tag{3}$$

In order to maximize the objective function, the sequence of decision vectors, θ^n, n = 1, 2,..., N must be so chosen that the following conditions are satisfied:

$$H^n = \sum_{i=1}^{2} z_i^n T_i^n(x^{n-1};\ \theta^n) = \text{maximum} \tag{4}$$

or

$$\frac{\partial H^n}{\partial \theta^n} = 0 , \qquad n = 1, 2, \ldots, N \tag{5}$$

where H^n is the Hamiltonian function for the n-th stage, and z^n is an adjoint vector given by the relationship

$$z_i^n = \frac{\partial H^n}{\partial x_i^{n-1}} , \qquad \begin{array}{l} n = 1, 2, \ldots, N \\ i = 1, 2 \end{array} \tag{6a}$$

and

$$z_i^N = c_i . \tag{6b}$$

For the one-dimensional process, the Hamiltonian function can be written as

$$H^n = z_1^n \, T(x_1^{n-1}; \theta^n) + z_2^n \{x_2^{n-1} + G(x_1^{n-1}; \theta^n)\} . \tag{7}$$

According to equation (6a), the recurrence relations for the adjoint variables, z_1 and z_2, are found to be

$$z_1^{n-1} = \frac{\partial T(x_1^{n-1}; \theta^n)}{\partial x_1^{n-1}} z_1^n + \frac{\partial G(x_1^{n-1}; \theta^n)}{\partial x_1^{n-1}} \tag{8a}$$

$$z_2^{n-1} = z_2^n , \qquad n = 1, 2, \ldots, N. \tag{8b}$$

From equation (3) it can be seen that $c_1 = 0$, $c_2 = 1$ and equation (6b) gives

$$z_1^N = 0 , \tag{8c}$$

$$z_2^N = 1 . \tag{8d}$$

Substituting equation (8d) into equations (8a) and (8b) gives

$$z_2^n = 1 , \qquad n = 1, 2, \ldots, N , \tag{9a}$$

$$z_1^{n-1} = \frac{\partial T(x_1^{n-1}; \theta^n)}{\partial x_1^{n-1}} z_1^n + \frac{\partial G(x_1^{n-1}; \theta^n)}{\partial x_1^{n-1}} , \quad n = 1, 2, \ldots, N. \tag{9b}$$

Hence the Hamiltonian function becomes

$$H^n = z_1^n T(x_1^{n-1};\ \theta^n) + G^n(x_1^{n-1};\ \theta^n) + x_2^{n-1},$$

$$n = 1, 2, \ldots, N .$$

According to equation (5),

$$\frac{\partial H^n}{\partial \theta^n} = z_1^n \frac{\partial T(x_1^{n-1};\ \theta^n)}{\partial \theta^n} + \frac{\partial G(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = 0.$$

Solving this equation for z_1^n yields

$$z_1^n = - \frac{\dfrac{\partial G(x_1^{n-1};\ \theta^n)}{\partial \theta^n}}{\dfrac{\partial T(x_1^{n-1};\ \theta^n)}{\partial \theta^n}} , \qquad n = 1, 2, \ldots, N . \tag{10}$$

The substitution of equation (10) into equation (9b) yields the recurrence relation

$$\frac{\dfrac{\partial G(x_1^{n-1};\theta^n)}{\partial \theta^n}}{\dfrac{\partial T(x_1^{n-1};\theta^n)}{\partial \theta^n}} = \frac{\dfrac{\partial G(x_1^n;\ \theta^{n+1})}{\partial \theta^{n+1}}}{\dfrac{\partial T(x_1^n;\ \theta^{n+1})}{\partial \theta^{n+1}}} \ \frac{\partial T(x_1^n;\ \theta^{n+1})}{\partial x_1^n} - \frac{\partial G(x_1^n;\ \theta^{n+1})}{\partial x_1^n} ,$$

$$n = 1, 2, \ldots, N-1. \tag{11}$$

Combination of equations (8c) and (10) yields

$$\frac{\partial G(x^{N-1};\ \theta^N)}{\partial \theta^N} = 0 . \tag{12}$$

The simultaneous use of the recurrence relation (11) and the performance equations (1) and (2) furnishes a powerful tool

in the optimization of those systems exhibiting the characteristics mentioned above.

For the solution of a problem with a prescribed end point, x_1^N, the condition of $z_1^N = 0$ (equation 8c) or the equivalent condition given by equation (12) is deleted.

A SIMPLE HEAT EXCHANGER TRAIN

In this section the application of the recurrence relation to the simple heat exchanger train will be demonstrated.

A schematic representation of a simple heat exchanger train (cross-current system) is shown in Figure 1. Each exchanger in the train is a counter current heat exchanger. A cold stream enters the first stage with a certain temperature $x_1^0 = a$, and leaves the final stage with a temperature $x_1^N = b$. The cold stream is heated at each stage by a hot stream counterflowing across the stage. The inlet and outlet temperatures of the hot stream flowing across the n-th stage are t_1^n and t_2^n respectively. Our discussion will be restricted to the case where WC_p, the products of fluid flow rate, W, and specific heat, C_p, are equal for all streams.

The problem is to select the area, θ^n, for each stage of the train so as to minimize the total area, with x_1^0, x_1^N, and t_1^n, $n = 1, 2, \ldots, N$ prescribed.

A heat balance at the n-th stage gives

$$WC_p(x_1^n - x_1^{n-1}) = WC_p(t_1^n - t_2^n)$$

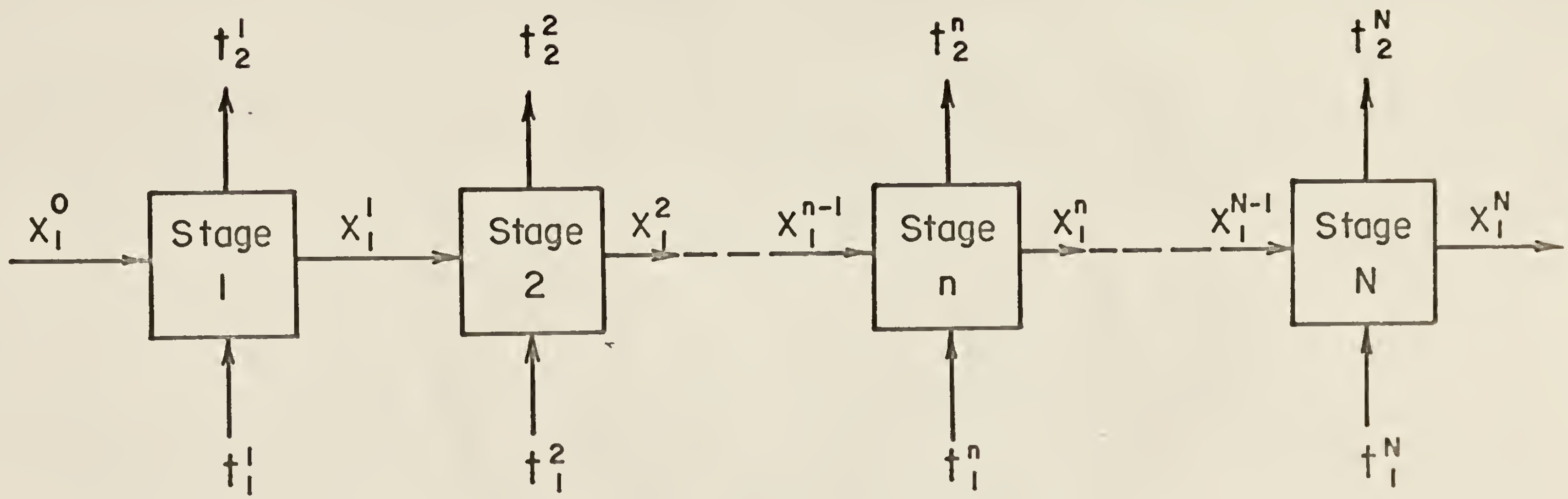


Fig. 1 Multistage heat exchanger.

or

$$t_1^n - x_1^n = t_2^n - x_1^{n-1} \tag{13}$$

which indicates that the temperature differences at the inlet and the outlet are equal. Equating the heat gain of the cold stream at the n-th stage to the heat transferred at the same stage gives

$$WC_p(x_1^n - x_1^{n-1}) = u^n \theta^n(t_1^n - x_1^n) \tag{14}$$

where u^n represents the overall heat transfer coefficient at the n-th stage.

Solving equation (14) for x_1^n, the following performance equation is obtained:

$$x_1^n = \frac{x_1^{n-1} + U^n t_1^n \theta^n}{1 + U^n \theta^n} \tag{15}$$

where

$$U^n = \frac{u^n}{WC_p} .$$

By introducing a new stage variable, x_2^n, satisfying the following performance equation and initial condition,

$$x_2^n = x_2^{n-1} + \theta^n , \qquad x_2^0 = 0 , \tag{16}$$

the problem is transformed into the standard form in which x_2^n is to be minimized by the proper selection of θ^n, n = 1, 2,..., N.

Comparing equations (15) and (16) with equations (1) and

(2), we obtain

$$T(x_1^{n-1};\ \theta^n) = \frac{x_1^{n-1} + U^n\ t_1^n\ \theta^n}{1 + U^n\ \theta^n} \tag{17}$$

and

$$G^n(x_1^{n-1};\ \theta^n) = \theta^n\ . \tag{18}$$

Differentiating equations (17) and (18) with respect to x_1^{n-1} and θ^n gives

$$\frac{\partial T(x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = \frac{1}{1 + U^n\ \theta^n}\ , \qquad n = 1, 2, \ldots, N, \tag{19}$$

$$\frac{\partial T(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = \frac{U^n(t_1^n - x_1^{n-1})}{(1 + U^n\ \theta^n)^2}\ , \qquad n = 1, 2, \ldots, N, \tag{20}$$

$$\frac{\partial G^n(x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 0\ , \qquad n = 1, 2, \ldots, N, \tag{21}$$

$$\frac{\partial G^n(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = 1\ , \qquad n = 1, 2, \ldots, N. \tag{22}$$

Substituting equations (19) through (22) into the recurrence relation (11) yields

$$\frac{(1 + U^n\theta^n)^2}{U^n(t_1^n - x_1^{n-1})} = \frac{1 + U^{n+1}\ \theta^{n+1}}{U^{n+1}(t_1^{n+1} - x_1^n)}\ . \tag{23}$$

Solving equation (14) for $U^n\theta^n$ and substituting the resulting expression into equation (23) gives

$$x_1^{n-1} = x_1^n + (x_1^n - t_1^n)\left\{\frac{U^n(x_1^n - t_1^n)}{U^{n+1}(x_1^{n+1} - t_1^{n+1})} - 1\right\}, \tag{24}$$

$$n = 1, 2, \ldots, N-1.$$

COMPUTATIONAL PROCEDURE

I. <u>Computational procedure based on an estimate of x_1^{N-1}</u>.

Since $x_1^N = b$ is fixed, we can start the computations by assuming a value for x_1^{N-1} to obtain x_1^{N-2}, x_1^{N-3},..., x_1^0 from equation (24). But before we do so, let us ascertain the range of possible values that x_1^{N-1} may take.

From simple observation we obtain x_1^0 as the lower bound for the range of x_1^{N-1}, and x_1^N as the upper bound. Therefore

$$x_1^0 \leq x_1^{N-1} \leq x_1^N. \tag{25}$$

But this range can be further reduced by a simple analysis of the characteristics of a counter-current heat exchanger.

The assumption that no change of phase occurs in the cold or in the hot stream at any stage is implied in equations (13) and (14). Figure 2 represents the counter-current heat exchange process across the n-th stage.

From Figure 2 we obtain

$$x_1^{n-1} - x_1^n < 0, \qquad n = 1, 2, \ldots, N \tag{26}$$

and

$$x_1^n - t_1^n < 0, \qquad n = 1, 2, \ldots, N. \tag{27}$$

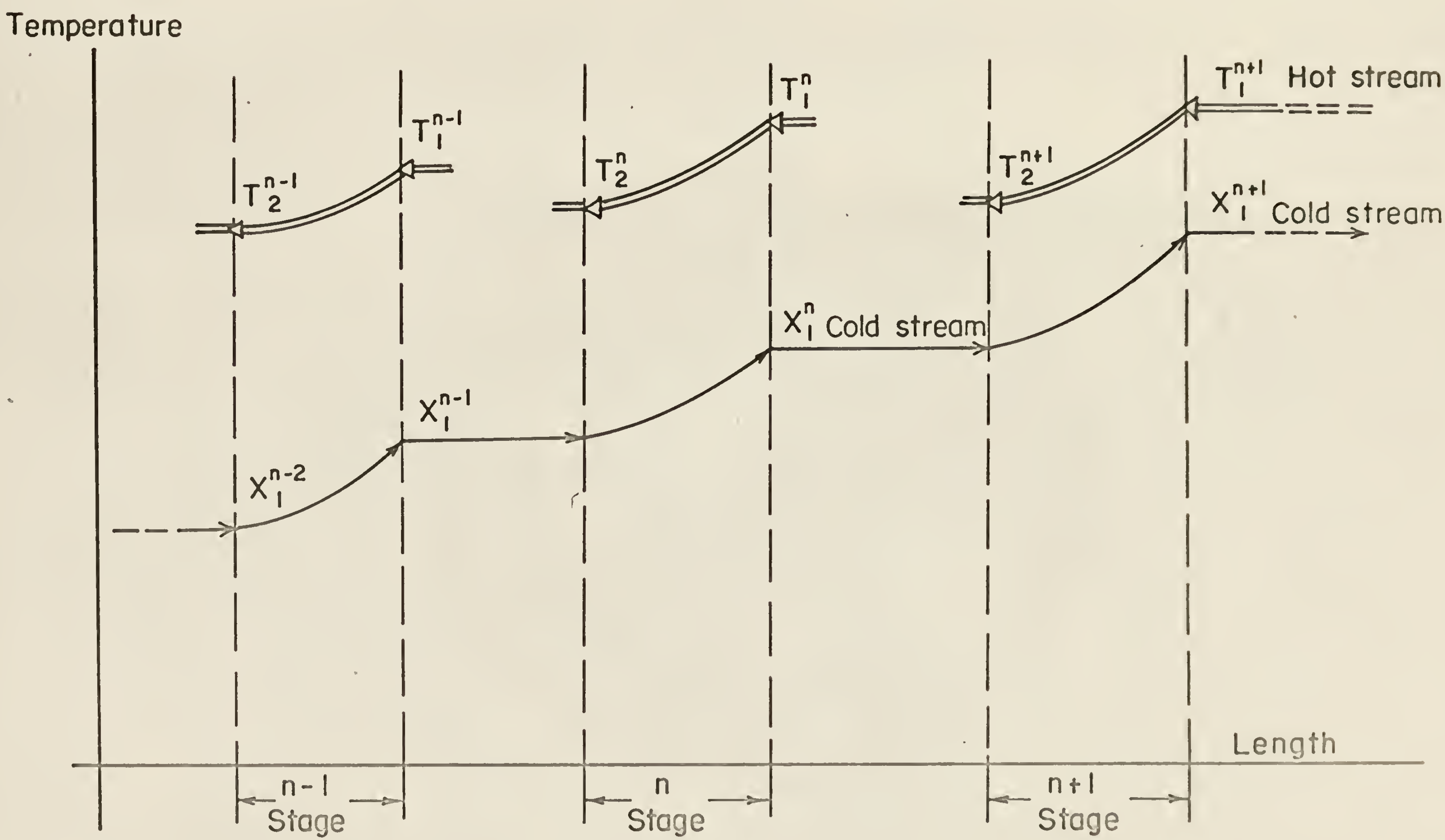


Fig. 2 Temperature distribution in a constant-phase counterflow multistage heat exchange process.

Equation (24) yields

$$x_1^{n-1} - x_1^n = (x_1^n - t_1^n)\left\{\frac{U^n(x_1^n - t_1^n)}{U^{n+1}(x_1^{n+1} - t_1^{n+1})} - 1\right\} < 0 \tag{28}$$

and by virtue of equation (27),

$$\left\{\frac{U^n(x_1^n - t_1^n)}{U^{n+1}(x_1^{n+1} - t_1^{n+1})} - 1\right\} > 0\ . \tag{29}$$

Solving equation (29) for x_1^N and letting $n = N-1$ we obtain, by virtue of equation (27),

$$x_1^{N-1} < \frac{U^N}{U^{N-1}}(x_1^N - t_1^N) + t_1^{N-1}\ . \tag{30}$$

Thus, the restricted range for x_1^{N-1} is finally obtained as

$$x_1^0 < x_1^{N-1} \leq \frac{U^N}{U^{N-1}}(x_1^N - t_1^N) + t^{N-1}. \tag{31}$$

Therefore, we may carry out the computational procedure as follows:

(i) Assuming a value for x_1^{N-1} and through the use of the recurrence equation, equation (24), we obtain the value of x_1^0.

(ii) If the given and the calculated values for x_1^0 are close enough (within an error bound), we accept the sequence of values for x_1^n as the optimal one and evaluate θ^n for each stage from equation (14).

(iii) If the given and the calculated values of x_1^0 differ significantly, we assume a new value for x_1^{N-1} and repeat the process.

It should be mentioned at this point that the sequence of

values obtained for each assumed value of x_1^{N-1} is by itself an optimal path between x_1^N and the computed value for x_1^0. The data so obtained, therefore, can be listed for possible design changes in the end conditions of the system.

Prior to our systematic search for values of x_1^{N-1}, we must define our permissible error limit, E_m, between the given and the calculated values of x_1^0 so as to satisfy the relation:

$$|E| = |(x_1^0)_{\text{given}} - (x_1^0)_{\text{calculated}}| \leq E_m, \tag{32}$$

where $E_m > 0$.

The calculation process then becomes a search for a value of x_1^{N-1} which generates a value for $(x_1^0)_{\text{calculated}}$ satisfying equation (32). This assumed value for x_1^{N-1} is actually an approximation to the root, or roots, of some error function

$$E = E(x_1^{N-1}) \tag{33}$$

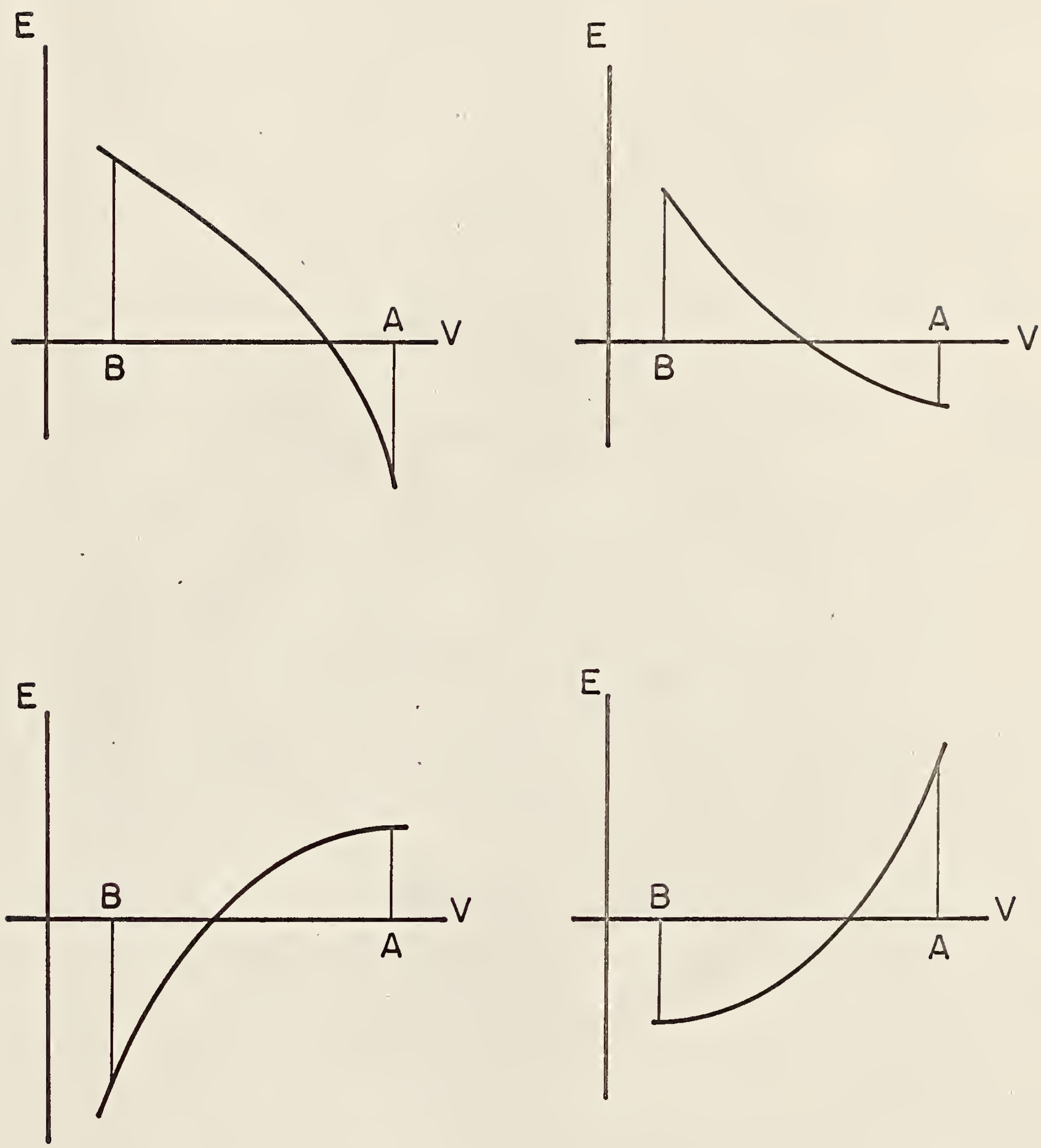
so that

$$|E| \leq E_m . \tag{34}$$

The four possible patterns that the error function may take in the vicinity of a root are depicted in Figure 3. Several techniques are available for obtaining an approximation to the root of the error function. In the following iterative procedure we will utilize the regular Falsi method in accordance with the nomenclature of Figure 4.

Iterational Procedure

Step 1. Assume a value for x_1^{N-1} equal to the upper bound of its range as given by equation (30).



V = Estimated state variable

Fig. 3 Four possible behaviors of the error function plotted vs. the state variable being estimated.

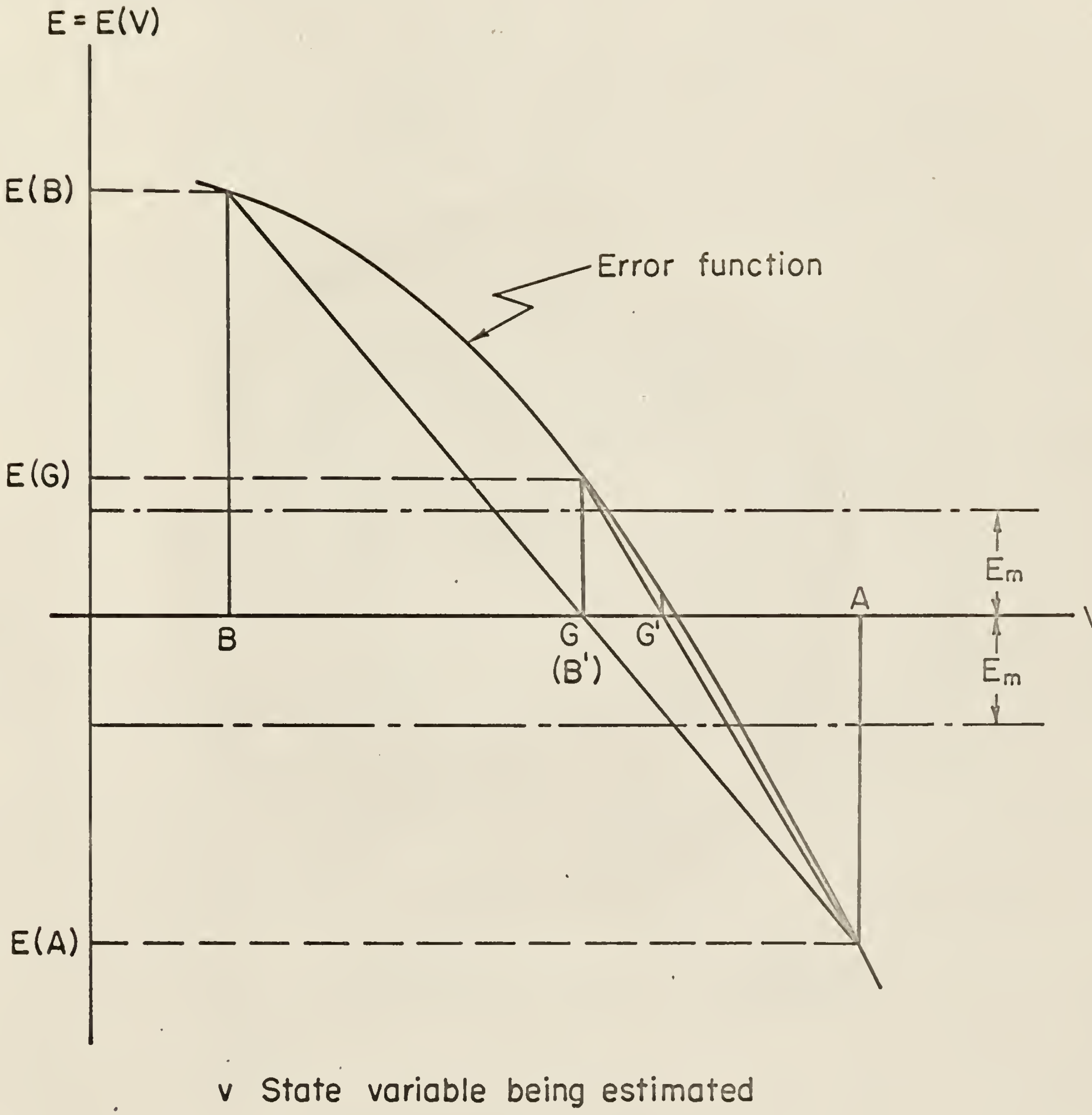


Fig. 4 Illustration of the parameters used in the regular falsi method.

Step 2. Calculate x_1^{N-2}, x_1^{N-3},..., x_1^0 from equation (24).

Step 3. If the error limit is satisfied as shown in equation (32), the sequence of x_1^n, n = N-2, N-3,..., 0 so obtained is the optimal sequence of heat exchanger temperatures in the train.

Step 4. Compute the optimum sequence of heat exchanger areas, θ^n, from equation (14).

Step 5. If the error limit is not satisfied, decrease x_1^{N-1} by D_a where

$$D_a = \frac{x_1^N - x_1^0}{4.5\ N}$$

and repeat Step 2 until the error limit is satisfied or a change in the sign of the error function occurs.

Step 6. When a change of sign in the error function occurs, enter the regular Falsi iterative process as follows:

a) Record the last two values given to x_1^{N-1}, say A and B in that order, and the corresponding values of the errors, E(A) and E(B).

b) Find the straight-line interpolation point, G, between the last two points, A and B, determined as follows:

$$G = \frac{A|E(B)| + B|E(A)|}{|E(B)| + |E(A)|}. \tag{35}$$

c) Let x_1^{N-1} = G and compute x_1^0 from equation (24).

d) Compute $E(G) = (x_1^0)_{given} - (x_1^0)_{calculated}$.

e) If E(G) meets the error limit, the sequence of x_1^n, n = N-2, N-3,..., 0 calculated in (c) is the

optimal path; then compute θ^n, $n = 1, 2, \ldots, N$ from equation (14).

f) If E(G) does not satisfy the error limit, proceed as follows: If E(G) and E(A) have like signs, let A = G. If E(G) and E(A) do not have like signs, let B = G.

g) Repeat steps (b) through (f) until the error limit is satisfied.

II. Computational Procedure Based on an Estimate of x_1^1.

In section I we derived a computational procedure in which the optimal sequence of stage temperatures, x_1^n, and heat exchanger areas, θ^n, for the N stages were calculated based on the trial estimates of x_1^{N-1}. In this section we shall derive a similar procedure in which the computation of the optimal path will be based on a trial estimate of the temperature, x_1^1.

From equation (24) we obtain

$$x_1^{n-1} = \frac{U^n}{U^{n+1}} \frac{(x_1^n - t_1^n)^2}{(x_1^{n-1} - t_1^n)} + t_1^{n+1} , \qquad n = 1, 2, \ldots, N. \tag{36}$$

Since $x_1^0 = a$ is fixed, we may start the calculations by assuming a value for x_1^1 and obtain x_1^n, $n = 2, 3, \ldots, N$ from equation (36).

The upper bound for the range of possible values of x_1^1 can be obtained directly from equation (27) by letting n=1. The lower bound is x_1^0. Therefore,

$$x_1^0 < x_1^1 < t_1^n .$$

The error limit is now defined as

$$|E| = |(x_1^N)_{given} - (x_1^N)_{calculated}| \leq E_m \tag{37}$$

where E_m is, again, the maximum allowable error. The error function becomes

$$E = E(x_1^1) \tag{38}$$

so that

$$|E| \leq E_m \ .$$

<u>Iterative Procedure</u>

Step 1. Assume a value for x_1^1 equal to the lower bound of its range, x_1^0.

Step 2. Calculate $x_1^2, x_1^3, \ldots, x_1^N$ from equation (36).

Step 3. If the error limit is satisfied as in equation (37), the sequence of x_1^n thus obtained is the optimal sequence of heat exchanger temperatures.

Step 4. Compute the optimum sequence of heat transfer areas, θ^n, from equation (14).

Step 5. If the error limit is not satisfied, <u>increase</u> x_1^1 by D_b, where

$$D_b = \frac{t_1^N - x_1^0}{4.5\ N}$$

and repeat step 2 until the error limit is satisfied, or a change in the sign of the error function occurs.

Step 6. When a change in the sign of the error function occurs, enter the regular Falsi iterative process. The same process presented in step 6 of section I is applied here with the only change being that x_1^{N-1} of section I should

be replaced by x_1^1, and step (d) should be changed to read: Compute

$$E(G) = (x_1^N)_{given} - (x_1^N)_{calculated}.$$

AN EXAMPLE

In order to illustrate the computational procedure, we shall consider a simple example of a heat exchanger train. The data for this example are shown in Table 1 (1).

The computations were carried out on an IBM 1620 computer. The FORTRAN program is included in Table 2 and a symbol table for this program is given in Table 3 of Appendix I.

Table 1. Data of Heat Exchanger Problem

$WC_p = 100{,}000$

$x_1^0 = 100°F$

$x_1^3 = 500°F$

Stage, n	u^n, BTU/(hr)(sq ft)(°F)	t_1^n, °F
1	120	300
2	80	400
3	40	600

TABLE A3 RESULTS

STAGE	EXIT TEMP.	STAGE AREA	
	OPTIMAL DESIGN	FOR X 0 =	225.000
		AND X 3 =	500.000
1	350.000	-2083.333	
2	350.000	0.000	
3	500.000	3750.000	
	TOTAL AREA=	1666.667	
	OPTIMAL DESIGN	FOR X 0 =	286.453
		AND X 3 =	500.000
1	273.182	-412.360	
2	320.370	740.741	
3	500.000	4490.741	
	TOTAL AREA=	4819.121	
	OPTIMAL DESIGN	FOR X 0 =	35.692
		AND X 3 =	500.000
1	161.248	754.080	
2	290.741	1481.482	
3	500.000	5231.482	
	TOTAL AREA=	7467.043	
	OPTIMAL DESIGN	FOR X 0 =	132.023
		AND X 3 =	500.000
1	193.302	478.605	
2	298.339	1291.518	
3	500.000	5041.518	
	TOTAL AREA=	6811.641	
	OPTIMAL DESIGN	FOR X 0 =	102.589
		AND X 3 =	500.000
1	182.903	571.564	
2	295.813	1354.667	
3	500.000	5104.667	
	TOTAL AREA=	7030.897	
	OPTIMAL DESIGN	FOR X 0 =	100.014
		AND X 3 =	500.000
1	182.023	579.264	
2	295.602	1359.942	
3	500.000	5109.942	
	TOTAL AREA=	7049.147	
	OPTIMAL DESIGN	FOR X 0 =	100.001
		AND X 3 =	500.000
1	182.018	579.304	
2	295.601	1359.969	
3	500.000	5109.969	
	TOTAL AREA=	7049.242	

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2. Fan, L. T., C. L. Hwang and C. S. Wang, "Optimization of Multistage Heat Exchanger Systems by the Discrete Maximum Principle," Chemical Engineering Progress Symposium, Ser. 59, Vol. 61, pp. 243-252, 1965.

4. SALES RESPONSE TO ADVERTISING

INTRODUCTION

The analytical study of promotional efforts has been advanced by a good number of investigators. Mathematical models of varying complexity have also been proposed, some of which are elegantly summarized and discussed in reference (1). Little, however, has been done in the area of analytical study of sales promotion through advertising. It is not our intention to present in this paper a new model for the optimization of sales promotions, but rather to demonstrate the applicability of the maximum principle to this type of management problem.

The mathematical model we shall occupy ourselves with was originally proposed by Vidale and Wolfe (4). The various parameters of this model are discussed first and then the model is optimized by the maximum principle. Two response functions similar to those suggested by Zentler and Ryde (5) are then incorporated into the original model in order to remove the linearity constraint imposed on the response function. And, finally, the modified system is optimized by the maximum principle.

ADVERTISING PARAMETERS

In order to measure the response of individual products in advertising, Vidale and Wolfe performed a large number of controlled experiments on actual advertising campaigns. Their description of the interaction between sales and advertising is based on three parameters:

1. λ, The sales decay constant
2. M, The saturation level
3. $\gamma[A(t)]$, The response function.

Sales Decay Constant. In the absence of advertising, the volume of sales tends to decrease due to customers abandoning the product because of obsolescence, product acceptability, competing advertising and like factors. This decrease, in general, appears to be constant and exponential in character regardless of market size. Furthermore, this decay effect persists even when advertising campaigns are being undertaken, but the decay is counterbalanced by a larger number of new customers buying the product.

Saturation Level. Under real conditions, it is logical to assume that not everybody will buy a given product even if he has learned about the product. The reason for this may be found on the well established affiliation of the potential new customer with a competing product or, in a broader case, the consumer's dissatisfaction after purchasing the new product. The net effect of this is a ceiling of the possible sales volume, or a saturation level.

Response Function. Of the three parameters in the model, this is perhaps the most difficult to visualize. The response function is defined as the sales generated per advertising dollar independently of the sales level. This response function, however, affects only that sector of the market which is not already buying the product. Regardless of the response function, therefore, the increments in sales obtained from each additional dollar spent on advertising becomes smaller as sales approach the

saturation level.

The value of each one of these parameters differs from product to product and must therefore be determined separately for individual items. The sales decay constant may be calculated from the variations observed in sales volume either after or before a sales promotion. The saturation level may be estimated from a market survey or past sales data on substitutive goods. The response function is indirectly affected by psycho-sociological factors and may be determined from past results obtained in the advertising media considered.

It was found by Vidale and Wolfe (4) that carefully designed test promotions at a sufficiently large scale give significant and reproducible results. Since product advertising, when effective, generates positive results within days or few weeks, advertising campaigns for new products may be pretested and the parameters estimated. As the campaign progresses, these estimates may be improved and adjusted towards the parameters of the actual campaign.

The purchasing response to any level of advertising expenditure is assumed to obey a deterministic function throughout the planning horizon. Similarly, the quality of advertising as well as the effectiveness of the advertising media used are assumed constant and totally determined by the values of the parameters discussed above. Under these conditions, the time rate of advertising expenditures becomes solely responsible for the optimization of the sales volume.

THE MATHEMATICAL MODEL

On the basis of the three parameters explained above, the change in the rate of sales, without advertising, is represented by:

$$\frac{dQ(t)}{dt} = -\lambda\, Q(t) \tag{1}$$

where Q(t) is the rate of sales at time t, in dollars per unit of time.

When advertising is introduced, equation (1) is transformed by admitting a positive factor which accounts for the effect of advertisement. Equation (1) becomes

$$\frac{dQ(t)}{dt} = -\lambda Q(t) + \gamma[A(t)]\left[1 - \frac{Q(t)}{M}\right] \tag{2}$$

where A(t) = rate of advertising at time t

M = the saturation level of sales for the product.

Both are given in dollars per unit of time.

It may be mentioned at this point that the introduction of advertising may change the probability of customers leaving the product and may also alter the overall market conditions. This means that additional parameters should be introduced into the mathematical model to account for the second order effects. But the degree of accuracy attained may not justify the additional complexity presented by the adjusted model and we shall limit our analysis to the model given by equation (2).

Notice that $(1-Q(t)/M)$ in equation (2) represents that fraction of the total market, M, which is not already purchasing the product. Consequently, this is the only portion of the market which is influenced by the advertising effect, $\gamma[A(t)]$.

If a constant rate of advertising, A, is assumed, the

following general solution for equation (2) is obtained

$$Q(t) = \left[\frac{M}{(1 + \frac{\lambda M}{\gamma A})}\right]\left\{ 1 - e^{-(\gamma A/M+\lambda)t}\right\} + Q_o e^{-(\frac{\gamma A}{M} + \lambda)t}, \qquad t < t_r, \tag{3}$$

$$Q(t) = Q(t_r)e^{-\lambda(t - t_r)}, \qquad t \geq t_r, \tag{4}$$

where Q_o is the rate of sales at time t=0 and t_r is some unspecified time during which a constant rate, A, of advertising expenditure is maintained. Figure 1 is a graphical representation of a sales response to an advertising campaign of duration t_r.

OPTIMIZATION OF THE MODEL

We shall now demonstrate the applicability of the maximum principle in the optimization of the model discussed above. Two main cases are considered in which the response to advertising, $\gamma[A(t)]$, is first: considered to change linearly with the advertising expenditure, and then an exponential variability is discussed. In both cases the optimization criteria will be the net revenue after advertising costs are discounted. Manufacturing costs and advertising costs are assumed to be independent. The optimization criteria is represented by the equation

$$S = \int_o^T [Q(t) - A(t)]\ dt \tag{5}$$

where T is the planning period. The problem, therefore, becomes that of determining an advertising expenditure function so that maximum revenues from sales as given by equation (5) are attained.

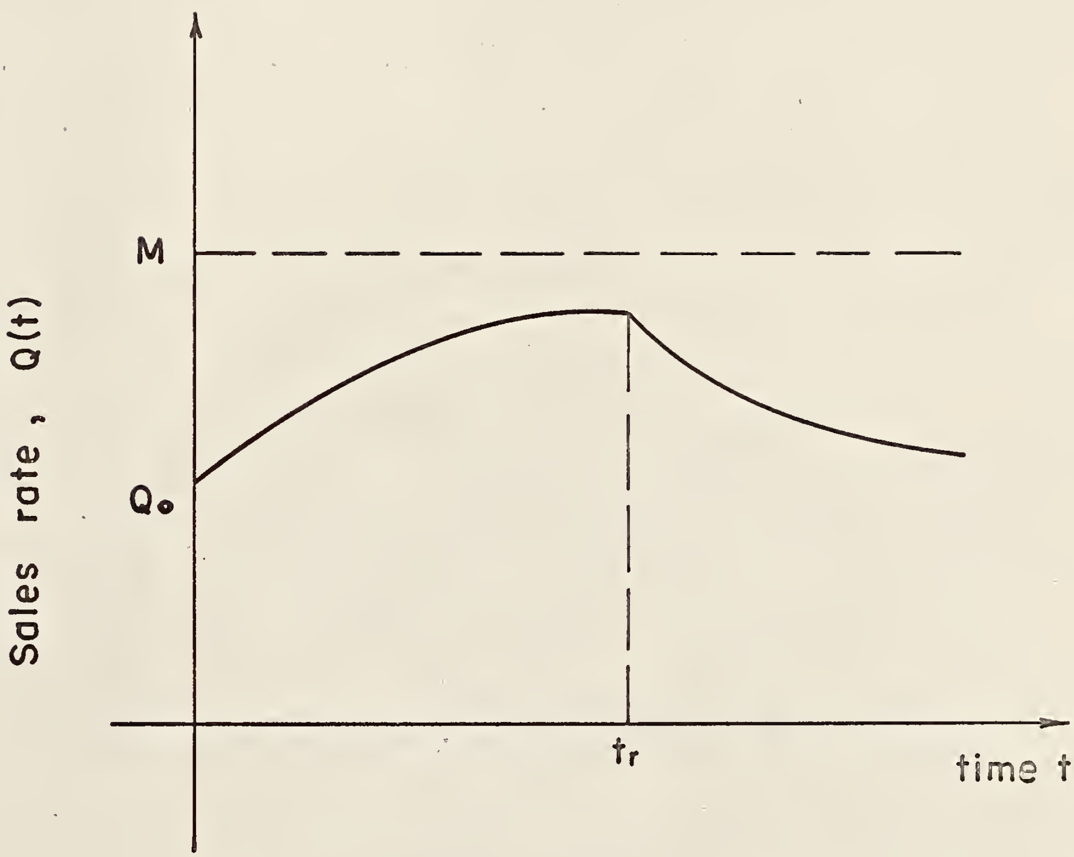

Fig. 1 Sales response to advertising.

FIRST CASE: LINEAR RESPONSE

In order to apply the maximum principle, let us define

$$\theta(t) = A(t) , \qquad 0 \leq \theta(t) \leq \theta_{max}, \tag{6}$$

$$x_1(t) = Q(t), \qquad x_1(o) = Q_o, \tag{7}$$

$$\frac{dx_1}{dt} = - \lambda x_1 + \gamma\theta(1 - \frac{x_1}{M}), \tag{8}$$

$$x_2 = \int_o^t [x_1 - \theta]\, dt, \qquad x_2(t) = 0, \tag{9}$$

$$\frac{dx_2}{dt} = x_1 - \theta, \tag{10}$$

where θ_{max} is the maximum rate permissible for advertising expenditures.

$$S = c_1 x_2(T) + c_2 x_2(T) = x_2(T) , \tag{11}$$

therefore, $c_1 = 0$, $c_2 = 1$.

The Hamiltonian function and the adjoint variables can now be written as

$$H = z_1 \frac{dx_1}{dt} + z_2 \frac{dx_2}{dt}$$

$$= z_1[-\lambda x_1 + \gamma\theta(1 - \frac{x_1}{M})] + z_2[x_1 - \theta], \tag{12}$$

$$\frac{dz_1}{dt} = - \frac{\partial H}{\partial x_1} = -z_1[-\lambda - \frac{\gamma\theta}{M}] - z_2, \tag{13}$$

$$z_1(T) = c_1 = 0, \tag{14}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0, \tag{15}$$

$$z_2(T) = c_2 = 1. \tag{16}$$

Solving equations (15) and (16) for $z_2(t)$ we obtain

$$z_2(t) = 1, \qquad 0 \le t \le T. \tag{17}$$

Substituting equation (17) into (12) and separating terms we obtain

$$H = H^* + x_1(1 - z_1\lambda) \tag{18}$$

where

$$H^* = [z_1\gamma(1 - \frac{x_1}{M}) - 1]\theta \tag{19}$$

is the variable part, with respect to θ, of the Hamiltonian. It is now apparent from equation (19) that the optimal control associated with this problem is of the "On-Off" or "Bang-Bang" type. If we let h be the coefficient of θ in equation (19), it follows immediately from equation (19) that [2]

$$\overline{\theta} = \begin{cases} 0 & \text{if} \quad h(t) < 0 \\ \theta_{max} & \text{if} \quad h(t) > 0 \end{cases} \tag{20}$$

where $\bar{\theta}$ is the optimum decision policy which will maximize the objective function given by equation (5). We are now to find the switching time, t_s, at which h changes sign. That is

$$h(t_s) = 0 \tag{21}$$

From the conditions obtained in equation (20), it is seen that θ is not a continuous function of time and that it may take only one of the extreme values. For computational purposes, then, θ may be assumed to be a constant,

$$\theta = \rho \qquad \text{where} \qquad \rho = \begin{cases} 0 \\ \theta_{max} \end{cases} . \tag{22}$$

Substituting equation (22) into (13) and by virtue of equations (14) and (17) we obtain

$$z_1(t) = \frac{1}{\omega}\,(1 - e^{\omega(t - T)}) \tag{23}$$

where

$$\omega = \lambda + \frac{\gamma\rho}{M} . \tag{24}$$

Substituting equation (22) into equation (8), then, solving the differential equation, we obtain

$$x_1(t) = (Q_o - \frac{\gamma\rho}{\omega})e^{-\omega t} + \frac{\gamma\rho}{\omega} . \tag{25}$$

Substitution of equations (23) and (25) into h, the coefficient of θ in the equation (19), yields

$$h = \frac{\gamma}{\omega}(1 - e^{\omega(t-T)})[1 - \frac{(Q_o - \frac{\gamma\rho}{\omega})e^{-\omega t} + \frac{\gamma\rho}{\omega}}{M}] - 1. \tag{26}$$

Letting h = 0 in equation (26) and rearranging terms we get

$$\alpha e^{-\omega t_s} + \beta e^{\omega t_s} + C = 0, \tag{27}$$

where

$$\alpha = \frac{\gamma}{\omega M} \left(\frac{\gamma\rho}{\omega} - Q_o\right), \tag{28}$$

$$\beta = \left(\frac{\gamma^2\rho}{\omega^2 M} - \frac{\gamma}{\omega}\right) e^{-\omega T}, \tag{29}$$

$$C = \frac{\gamma}{\omega} - \frac{\gamma^2\rho}{\omega^2 M} + \frac{\gamma}{\omega M} \left(Q_o - \frac{\gamma\rho}{\omega}\right) e^{-\omega T} - 1. \tag{30}$$

Letting $Y = e^{-\omega t_s}$ in equation (27) and solving it for Y we obtain

$$Y = -\frac{C}{2\alpha} \pm \sqrt{\left(\frac{C}{2\alpha}\right)^2 - \frac{\beta}{\alpha}}$$

or,

$$t_s = -\frac{1}{\omega} \ln \left(- \frac{C}{2\alpha} \pm \sqrt{\left(\frac{C}{2\alpha}\right)^2 - \frac{\beta}{\alpha}}\right). \tag{32}$$

It follows immediately that the values attainable by t_s and, consequently, the advertising policy for the period T depend on the initial conditions and parametric values of the model. But from equation (26), regardless of any conditions, the value of h at t = T is always

$$h\ (T) = -1 < 0. \tag{33}$$

The conditions given in equation (33) leads us to only three possible advertising policies depending on the values, t_{s1}, and t_{s2}, attained by t_s in equation (32). These three policies are depicted graphically in figure 2.

(i) Policy One. If neither t_{s1} nor t_{s2} fall in the interval $0 \leq t \leq T$, or if both are imaginary, no advertising should be done during the period T. That is

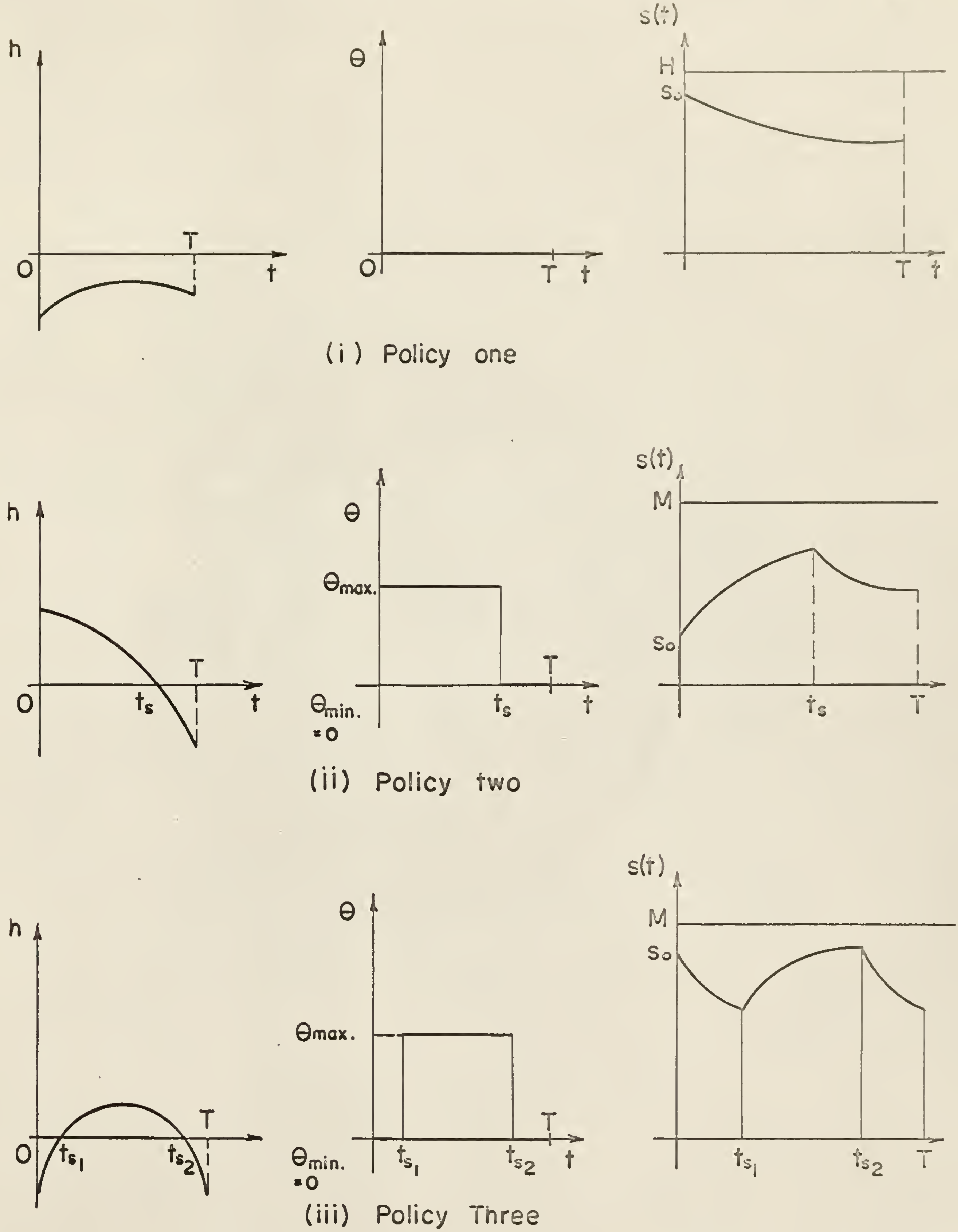

Fig. 2 Three main advertising policies.

$$\bar{\theta} = 0 \quad , \quad 0 \leq t \leq T. \tag{34}$$

This policy may be forced when the sales level is very close to the saturation level; the decay constant, λ, and/or the response constant, γ, are very small. Under these conditions, advertising becomes unprofitable.

(ii) Policy Two. If equation (32) generates a value of t_s which is real and falls inside the interval $0 \leq t \leq T$, then the optimal policy calls for the maximum constant rate of advertising during the first part of the period. That is

$$\bar{\theta} = \begin{cases} \theta_{max} & \text{for } 0 \leq t \leq t_s \\ \\ 0 & \text{for } t_s \leq t \leq T. \end{cases} \tag{35}$$

In this case, if A is the total advertising fund available for the period T, θ_{max} becomes

$$\theta_{max} = \frac{A}{t_s} . \tag{36}$$

It is easy to visualize in figure 2 that t_s is that time when sales approach the saturation level and only very small gains in sales are obtainable by further advertising. At this time advertising becomes uneconomical.

(iii) Policy Three. Under this case, both values of t_s given by equation (32) are real and satisfy the condition

$$0 \leq t_{s1} < t_{s2} \leq T. \tag{37}$$

For this case the optimum policy becomes

$$\bar{\theta} = \begin{cases} 0 & \text{for } 0 \leq t \leq t_{s1} \\ \theta_{max} & \text{for } t_{s1} \leq t \leq t_{s2} \\ 0 & \text{for } t_{s2} < t \leq T \end{cases} \tag{38}$$

where θ_{max} is, again, the maximum constant rate of advertising possible and is given by

$$\theta_{max} = \frac{A}{t_{s2} - t_{s1}}. \tag{39}$$

A special case occurs when $t_{s1} = t_{s2}$ renders h always a negative or zero value. Under this condition, policy one applies.

SECOND CASE: EXPONENTIAL RESPONSE

It was assumed in the basic model developed by Vidale and Wolfe that the response function, $\gamma[A(t)]$, increases linearly with the rate of advertising regardless of the sales saturation level, M. Under actual conditions, however, due to factors such as competing advertising or communication effectiveness, one might expect to find a saturation level for the response function beyond which no increase in the advertising effect can be achieved regardless of any increase on advertising expenditures.

The same effect on the response to advertising has been supported by Zentler and Ryde (5) while advertising under competition with a substitutive commodity. In order to correlate promotional activity and the response to this activity, Zentler and Ryde introduced an S-shaped curve which embodies the following ideas: "when promotion is at first started, the response is very small, but, once the required 'softening up' process has been performed there is a range in which response rises rapidly as promotional activity increases. Ultimately, as promotion is increased to much higher levels, the rate of increase in response tails off again and a point is reached at which further promotion produces very little additional effect".

Although the algebraic form of the family of curves suggested by Zentler and Ryde is rather complex, the basic shape or behavior of the response function can be closely reproduced by using exponential functions. In the analysis that follows, two basic exponential functions will be introduced in an attempt to reflect more realistically the response characteristics of a competitive market.

The two exponential functions are treated separately, but the assumption that the advertising effect influences only that sector of the market not already purchasing the product will be maintained under both conditions.

Exponential Functions. The two exponential functions are depicted graphically in figure 3. They can be written

$$\gamma[A(t)] = Ke^{-\frac{r}{A(t)}}, \tag{40}$$

and

$$\gamma[A(t)] = K(1 - e^{-rA(t)}).$$

These two functions display the properties that without advertisement expenditure ($A(t) = 0$) the advertisement response is null, and that the response to advertising does not increase linearly but exponentially as expenditures on advertising campaigns increase. In both cases the response function approaches assymtotically a saturation level K which for practical purposes may be identified with the market capacity or sales saturation level M. The two functions, however, differ basically in their behavior at low levels of advertising expenditures.

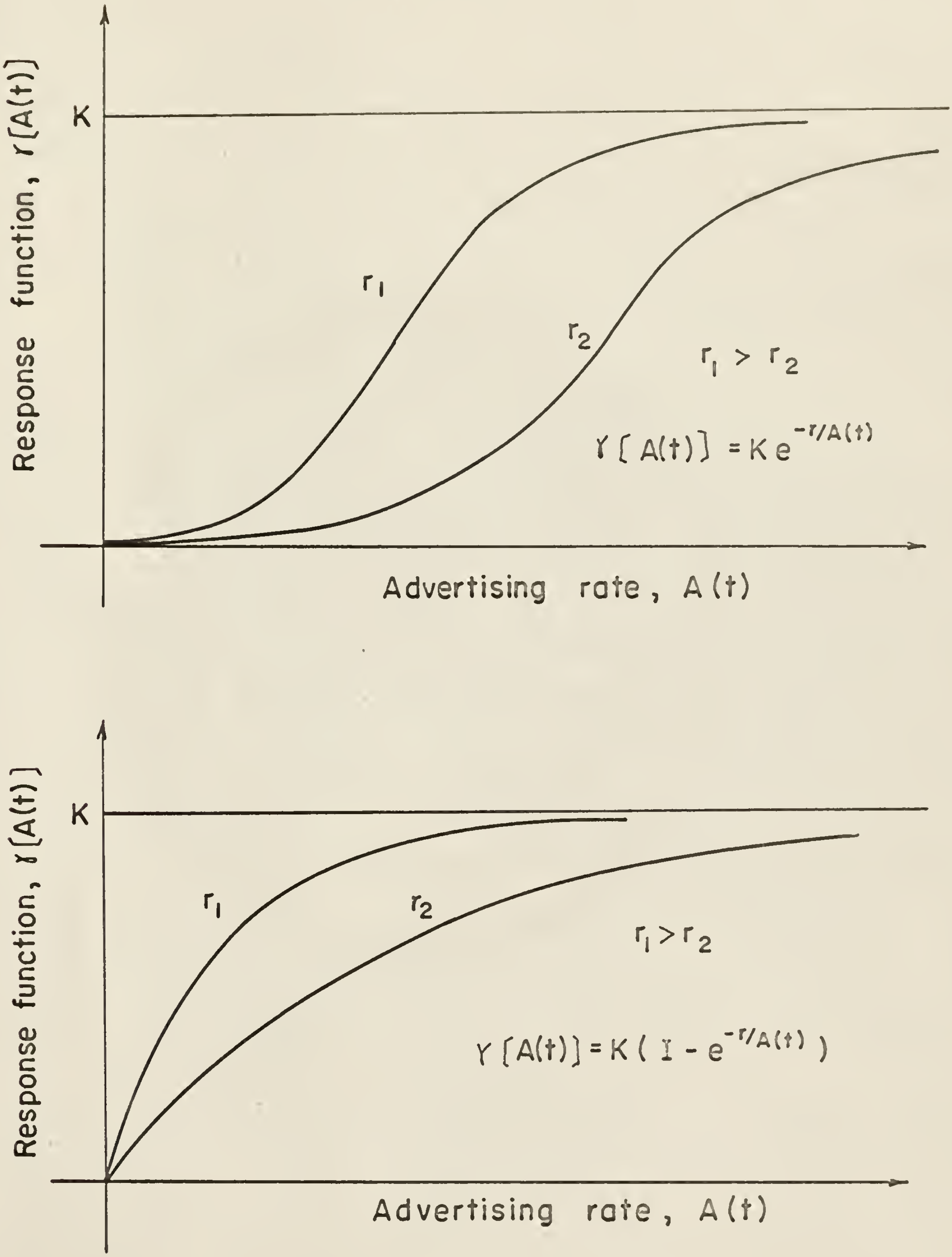

Fig. 3 Exponential functions.

The factor r determines the rapidity with which the response function rises and approaches the saturation level K. The determination of the value of r must rely on data from advertising campaigns done in the past for related products under similar market conditions. In addition to the parameter r, actual conditions may be approximated more closely by the proper choice of the exponential function.

Let us now discuss the application of these two functions to the basic advertising model.

Function one. To apply the maximum principle, let us again define

$$\theta(t) = A(t) \quad , \quad 0 \le \theta(t) \le \theta_{max} \tag{42}$$

where θ_{max} is the maximum permissible rate of advertising. It is determined from the condition that the total advertising expenditure for the period T does not exceed the available or allocated fund. Let

$$x_1(t) = Q(t) \quad , \quad x_1(0) = Q_o, \tag{43}$$

$$\begin{aligned} \frac{dx_1}{dt} &= -\lambda x_1 + \gamma[\theta(t)]\,(1 - \frac{x_1}{M}) \\ &= -\lambda x_1 + Ke^{-\frac{r}{\theta}}\,[1 - \frac{x_1}{M}], \end{aligned} \tag{44}$$

$$x_2(t) = \int_o^t [x_1 - \theta]dt, \quad x_2(0) = 0, \tag{45}$$

$$\frac{dx_2}{dt} = x_1 - \theta. \tag{46}$$

The objective function as given by equation (5) can be written

$$S = c_1x_1(T) + c_2x_2(T) = x_2(T), \tag{47}$$

therefore, $c_1 = 0$, $c_2 = 1$.

The Hamiltonian function and adjoint variables are

$$H = z_1[-\lambda x_1 + Ke^{-\frac{r}{\theta}}(1 - \frac{x_1}{M})] + z_2[x_1 - \theta], \tag{48}$$

$$\frac{dz_1}{dt} = - \frac{\partial H}{\partial x_1} = z_1[\lambda + \frac{K}{M}e^{-\frac{r}{\theta}}] - z_2, \tag{49}$$

$$z_1(T) = C_1 = 0,$$

$$\frac{dz_2}{dt} = - \frac{\partial H}{\partial x_1} = 0 \quad ,$$

$$z_2(T) = C_2 = 1. \tag{50}$$

From equation (50) we obtain

$$z_2(t) = 1 \quad , \quad 0 \leq t \leq T. \tag{51}$$

Substituting equation (51) into equation (48) and separating terms in the Hamiltonian function we obtain

$$H = H^* + x_1(1 - \lambda z_1), \tag{52}$$

where

$$H^* = z_1Ke^{-\frac{r}{\theta}}(1 - \frac{x_1}{M}) - \theta \tag{53}$$

is the variable part of the Hamiltonian.

Applying the optimality condition $\frac{\partial H}{\partial \theta} = 0$ to equation (53) we obtain

$$rKz_1(1 - \frac{x_1}{M})e^{-\frac{r}{\overline{\theta}}} = \overline{\theta}^2. \tag{54}$$

Equation (54) does not give the optimum decision, $\overline{\theta}(t)$, as an explicit function of time and, consequently, we must also solve simultaneously for z_1 and x_1. The set of differential equations involved is highly non-linear and the process calls for a

numerical solution of the system of equations given below:

$$\frac{d\bar{x}_1}{dt} = -\lambda\bar{x}_1 + K(1 - \frac{\bar{x}_1}{M})e^{-\frac{r}{\bar{\theta}}} \quad , \quad x_1(0) = Q_o \tag{55}$$

$$\frac{d\bar{z}_1}{dt} = \bar{z}_1[\lambda + \frac{K}{M}e^{-\frac{r}{\bar{\theta}}}] - 1 \quad , \quad z_1(T) \tag{56}$$

$$\bar{z}_1(1 - \frac{\bar{x}_1}{M})rKe^{-\frac{r}{\bar{\theta}}} = \bar{\theta}^2. \tag{57}$$

where $\bar{x}_1(t)$, $\bar{z}_1(t)$ and $\bar{\theta}(t)$ are the optimum functions $x_1(t)$, $z_1(t)$ and $\theta(t)$ respectively which will maximize the objective function as given by equation (5).

The optimum advertising rate, then, is given by

$$A(t) = \bar{\theta}(t). \tag{58}$$

The advertising rate as given by equation (58) will generate an optimum sales function

$$Q(t) = \bar{x}_1(t). \tag{59}$$

<u>Function</u> <u>two</u>. Letting

$$\gamma[A(t)] = K(1 - e^{-rA(t)}) \tag{60}$$

and following a process similar to that presented in the treatment of function one we obtain

$$\frac{dx_1}{dt} = -\lambda x_1 + K(1 - e^{-r\theta})(1 - \frac{x_1}{M}) \tag{61}$$

where θ is, again, the advertising rate and $x_1(t)$ is the sales function,

$$\frac{dx_2}{dt} = x_1 - \theta \quad , \quad x_2(0) = 0 \tag{62}$$

The objective function becomes

$$S = c_1x_1(T) + c_2x_2(T) = x_2(T). \tag{63}$$

The Hamiltonian can now be written as

$$H = z_1\{-\lambda x_1 + K(1 - e^{-r\theta})(1 - \frac{x_1}{M})\} + z_2\{x_1 - \theta\} \tag{64}$$

from which we obtain

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = z_1[\lambda + \frac{K}{M}(1 - e^{-r\theta})] - z_2, \tag{65}$$

$$z_1(T) = C_1 = 0,$$

and

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0 \quad , \quad z_2(T) = C_2 = 1. \tag{66}$$

Equation (66) gives

$$z_2(t) = 1 \quad , \quad 0 \leq t \leq T. \tag{67}$$

Substituting equation (67) back into (64) and separating terms the Hamiltonian function becomes

$$H = H^* + x_1(1 - \lambda z_1) + z_1K(1 - \frac{x_1}{M}) \tag{68}$$

where the variable part, H^*, has the form

$$H^* = -z_1Ke^{-r\theta}(1 - \frac{x_1}{M}) - \theta. \tag{69}$$

Applying the optimality condition, $\frac{\partial H}{\partial \theta} = 0$, we obtain

$$Ke^{-r\theta} = \frac{1}{r\bar{z}_1(1 - \frac{\bar{x}_1}{M})} \tag{70}$$

from which we get

$$\theta = \frac{1}{r} \ln \{rKz_1(1 - \frac{x_1}{M})\}. \tag{71}$$

Substituting equations (67) and (70) into equations (61) and (65) we obtain

$$\frac{d\bar{z}_1}{dt} = p\bar{z}_1 - \frac{1}{r(M - x_1)} - 1, \tag{72}$$

$$\frac{d\bar{x}_1}{dt} = -p\bar{x}_1 - \frac{1}{r\bar{z}_1} + K, \tag{73}$$

where

$$p = \lambda + \frac{K}{M}. \tag{74}$$

The simultaneous solution of equations (72) and (73) gives the optimum functions for $x_1(t)$ and $z_1(t)$. Once these functions are known, the optimum advertising rate is directly obtained from equation (71).

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5. OPTIMUM PRODUCTION PLANNING

INTRODUCTION

It is a purpose of this thesis to demonstrate the applicability of the maximum principle in obtaining an optimum policy for production planning.

Although there are many approaches and models for production planning through time, none is universally best. The basic model presented here is that of Holt *et al*. (6). This model, a projection of a servomechanism to a dynamic inventory system, has been originally treated by the maximum principle by Hwang and Fan (7). Their original treatment of the model is presented first and then two modifications of the basic model are also treated by the maximum principle. Under the three cases presented here, the minimization of the total cost for the planning period will be the optimizing criteria.

The original model developed by Holt *et al*. does not take into consideration the costs associated with changes in the rate of production. It is customary in designing production criteria of the discrete type (1, 2, 3), however, to consider the costs of changing the production level from one period to the other. The two modifications of the basic model treated here, although of continuous character, incorporate these costs into the analysis.

THE ORIGINAL MODEL

Forecasting is used by manufacturing companies in order to design production rules which anticipate and prepare for sales

fluctuations. These forecasts are not always precise and a buffer inventory is maintained in order to damper abrupt fluctuations in sales which may cause runouts or may force rapid changes in the rate of plant operation.

In general, the rate of change in the inventory level is equal to the difference between the rate of production and sales rate, that is,

$$\frac{dI}{dt} = P(t) - Q(t) \tag{1}$$

where I(t), P(t), and Q(t) represent the inventory level, production rate and sales rate respectively.* Although the dynamic characteristics of a production scheduling system are dependent upon the relation between sales forecasts and actual sales, it is assumed here that sales are known with certainty, that is, Q(t) may be a known prescribed function of time, or a constant.

It is also assumed that the costs from holding inventories and/or stockouts will be approximated by the quadratic $C_I[I(t) - \overline{I}]^2$, whereas the rate at which manufacturing costs are incurred can be approximated by the quadratic $C_p[P(t) - \overline{P}]^2$, where C_I and C_p are constants, and $\overline{I}$ and $\overline{P}$ represent the desired inventory and the production level of the plant respectively. Both $\overline{I}$ and $\overline{P}$ may be functions of time t. For simplicity, however, both will be considered as constants. Therefore, the total cost incurred during the period between time 0 and time T can be written

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* I(t), P(t), and Q(t) may be given in \$/(unit time) or (physical units)/(unit of time). The units of C_I, C_p must be determined accordingly.

$$C_T = \int_o^T \{C_I[I(T) - \bar{I}]^2 + C_P[P(t) - \bar{P}]^2\}dt \qquad (2)$$

where T, some time in the future, is not necessarily the length of the season.

The problem, then, is to find the optimum production rate which minimizes the total cost function represented by equation (2) subject to the constraint given by equation (1). The objective function to be minimized can be written

$$S = \int_o^T \{C_I[I(t) - \bar{I}]^2 + C_P[P(t) - \bar{P}]^2\}dt \ . \qquad (3)$$

OPTIMIZATION OF THE MODEL

In order to apply the maximum principle, let us define

$$x_1(t) = I(t), \qquad (4)$$

and

$$\theta(t) = P(t), \qquad (5)$$

where $\theta(t)$ is the decision variable to be chosen. Then, equation (1) becomes

$$\frac{dx_1}{dt} = \theta(t) - Q(t) \quad , \quad x_1(0) = I_o \qquad (6)$$

where $Q(t)$ is a certain fixed function representing the sales forecast.

We introduce $x_2(t)$ such that

$$x_2(t) = \int_o^t \{C_I[x_1(t) - \bar{I}]^2 + C_P[\theta(t) - \bar{P}]^2\}dt \qquad (7)$$

and from equation (7) we obtain

$$\frac{dx_2}{dt} = C_I[x_1(t) - \bar{I}]^2 + C_P[\theta(t) - \bar{P}]^2, \; x_2(0) = 0. \tag{8}$$

The objective function becomes

$$S = c_1x_1(T) + c_2x_2(T) = x_2(T) \tag{9}$$

from which we obtain $c_1 = 0$, $c_2 = 1$.

The Hamiltonian function and the adjoint variables can be written

$$H(z_1,x_1,\theta) = z_1(\theta - Q) + z_2[C_I(x_1 - \bar{I})^2 + C_P(\theta - \bar{P})^2], \tag{10}$$

$$\frac{dz_1}{dt} = \frac{-\partial H}{\partial x_1} = - 2z_2C_I(x_1 - \bar{I}), \; z_1(T) = C_1 = 0, \tag{11}$$

$$\frac{dz_2}{dt} = \frac{-\partial H}{\partial x_2} = 0, \qquad z_2(T) = C_2 = 1. \tag{12}$$

Solving equation (12) for $z_2(t)$ gives

$$z_2(t) = 1 \quad , \quad 0 \leq t \leq T \; . \tag{13}$$

Substituting equation (13) into (10) gives

$$H(z_1,x_1,\theta) = z_1(\theta - Q) + C_I(x_1 - \bar{I})^2 + C_P(\theta - \bar{P})^2. \tag{14}$$

According to the maximum principle, the optimality condition $\frac{\partial H}{\partial \theta} = 0$ gives the optimal control for this problem. From equation (14) we obtain, then,

$$\frac{\partial H}{\partial \theta} = 0 = z_1 + 2C_P(\theta - \bar{P}) \tag{15}$$

or

$$z_1(t) = - 2C_P[\theta(t) - \bar{P}]. \tag{16}$$

The combination of equations (15) and (11) yields

$$- 2C_P \frac{d\theta}{dt} = -2C_I(x_1-\bar{I}) \tag{17}$$

or

$$C_I(x_1(t) - \overline{I}) - C_p \frac{d\theta}{dt} = 0. \tag{18}$$

The simultaneous solution of the pair of differential equations given by equations (18) and (6) yields the optimum inventory and production policies. These solutions are

$$x_1(t) = A_1e^{\lambda t} + A_2e^{-\lambda t} + (x_1)_p, \tag{19}$$

$$\overline{\theta}(t) = A_1\lambda e^{\lambda t} - A_2\lambda e^{-\lambda t} + \frac{d(x_1)_p}{dt} + Q(t), \tag{20}$$

where

$$\lambda = \sqrt{\frac{C_I}{C_p}}\ . \tag{21}$$

A_1 and A_2 are constants which may be determined from the initial conditions associated with the problem, and $(x_1)_p$ identifies the particular solution in equation (19) and which is decided by the form and/or the values of the functions $\overline{I}$ and S.

ADDITIONAL COST FACTORS

The model discussed above assumes the existence of a desired inventory level $\overline{I}$ and production rate $\overline{P}$. Any deviations from these levels are assumed to induce costs as given by equation (2). The model, therefore, accounts for those costs which are directly related with the size of the deviation from the desired levels, but does not take into consideration the rate at which these deviations are induced or diminished.

It is apparent that high inventory levels due to abrupt decreases in sales may be avoided by appropriately timed sharp

decreases in the production rate (8). It is also possible to prevent runouts due to sudden increases in the sales volume by providing rapid increases in the production rate. These measures, however effective, are accompanied by significantly high costs which are induced by factors such as labour force inertia, reordering and production scheduling, etc., and must therefore be taken into consideration when designing production policies.

Some analyses of discrete functional character [6] often use the quadratic

$$K_1 \; [P_i - P_{i-1}]^2$$

to identify the aggregate cost of changing the production rate from level P_i to level P_{i-1}, or vice-versa, during or after some discrete interval of time Δt. The average cost rate associated with these changes in the production level, then, can be written

$$K_2 \left[\frac{\Delta P}{\Delta t}\right]^2 = K_2 \left[\frac{P_i - P_{i-1}}{\Delta t}\right]^2$$

where K_1 and K_2 are some specified constants. Similarly, the rate at which these costs occur in the continuous case may be approximated by obtaining the limit,

$$K_2 \left[\frac{dP}{dt}\right]^2 = \lim_{\Delta t \to 0} K_2 \left[\frac{\Delta P}{\Delta t}\right]^2$$

This expression will be used in the analysis that follows.

We shall consider first a simplified model which removes the constraints imposed by the desired inventory and production levels, $\overline{I}$ and $\overline{P}$, but which takes into consideration the cost of changing the production rate. A more comprehensive model will then be studied.

SIMPLIFIED MODEL

Since no preference levels for inventory or production are given, the production rate will be directly regulated by a known (deterministic) sales forecast, Q(t), and the costs induced by approximating this forecast throughout the planning period T. Let us assume, therefore, that the rate of costs from holding inventories and/or stockouts will be approximated by the quadratic

$$C_A[Q(t) - P(t)]^2, \tag{22}$$

whereas the rate of costs associated with changes in the production rate is approximated by the quadratic

$$C_B \left[\frac{dP(t)}{dt}\right]^2, \tag{23}$$

where P(t) is the rate of production, and C_A and C_B are constants. Therefore, the total cost incurred between time 0 and T is

$$CT = \int_0^T \{C_A[Q(t) - P(t)]^2 + C_B \left[\frac{dP(t)}{dt}\right]^2\}dt, \tag{24}$$

where T, again, is not necessarily the length of the season.

The problem is that of finding the optimum production rate $\overline{P}(t)$ which will minimize the cost function as given by equation (24). The objective function, then, can be written

$$S = \int_0^T \{C_A[Q(t) - P(t)]^2 + C_B \left[\frac{dP(t)}{dt}\right]^2\}dt. \tag{25}$$

OPTIMIZATION OF THE MODEL

In order to apply the maximum principle, let us define

$$\theta(t) = P(t) \qquad , \qquad \theta(0) = P_o \tag{26}$$

where $\theta(t)$ is the decision variable to be determined, P_o is the initial production rate at time $t = 0$. Let us define a state variable $x_1(t)$ such that

$$x_1(t) = \int_o^t \{C_A(Q - \theta)^2 + C_B(\frac{d\theta}{dt})^2\}dt \tag{27}$$

from which we obtain

$$\frac{dx_1}{dt} = C_A(Q - \theta)^2 + C_B(\frac{d\theta}{dt})^2, \ x_1(0) = 0. \tag{28}$$

The system defined by equations (26) and (28) does not contain the standard form required by the maximum principle.* We must, therefore, standardize this system before the maximum principle can be applied. In order to perform this transformation, let us introduce an additional state variable

$$x_2(t) = \theta(t) \tag{29}$$

and a new decision variable such that

$$\omega = \frac{d\theta}{dt} . \tag{30}$$

Therefore,

$$\frac{dx_2}{dt} = \omega \qquad , \qquad x_2(0) = \theta(0) = P_o. \tag{31}$$

In terms of equations (29) through (31) $x_1(t)$ becomes

$$x_1(t) = \int_o^t \{C_A(Q - x_2)^2 + C_B(\omega)^2\}dt \tag{32}$$

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* This problem belongs to that category of systems containing memory in the decision (4).

and

$$\frac{dx_1}{dt} = C_A(Q - x_2)^2 + C_B(\omega)^2, \ x_1(0) = 0. \tag{33}$$

Equations (31) and (33) now constitute a standard system and the objective function can be written

$$S = c_1 x_1(T) + c_2 x_2(T) = x_1(T), \tag{34}$$

Therefore, $c_1 = 1$, $c_2 = 0$.

The Hamiltonian function becomes

$$H(x_1, z_1, \omega) = z_1\{C_A[Q - x_2]^2 + C_B[\omega]^2\} + z_2\{\omega\}. \tag{35}$$

From equation (35) we obtain

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = 0 \quad , \quad z_1(T) = c_1 = 1, \tag{36}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 2z_1[Q - x_2]C_A, \ z_2(T) = c_2 = 0. \tag{37}$$

Solving equation (36) for $z_1(t)$ we obtain

$$z_1(t) = 1 \quad , \quad 0 \leq t \leq T. \tag{38}$$

Hence, the Hamiltonian function can be written

$$H = C_A[Q - x_2]^2 + C_B[\omega]^2 + z_2\omega. \tag{39}$$

According to the maximum principle, the optimum decision function for this problem can be obtained from the optimality condition $\frac{\partial H}{\partial \omega} = 0$. Applying this condition to equation (39) we obtain

$$\frac{\partial H}{\partial \omega} = 0 = 2\omega C_B + z_2 \tag{40}$$

or

$$z_2 = -2\omega C_B. \tag{41}$$

Differentiating equation (41) with respect to time and substituting the result into equation (37) yields

$$\frac{d\omega}{dt} = -\frac{C_A}{C_B}[Q - x_2]. \tag{42}$$

$$\frac{d\omega}{dt} = - \frac{C_A}{C_B} [Q - x_2]. \tag{42}$$

Differentiating equation (31) and substituting for ω in equation (42) gives

$$\frac{d^2x_2}{dt^2} - \frac{C_A}{C_B} x_2 = - \frac{C_A}{C_B} Q. \tag{43}$$

The solution of equation (43) constitutes, by virtue of equation (40), the optimum production policy for the planning period. The solution of equation (43) yields

$$x_2(t) = A_1e^{\lambda t} + A_2e^{-\lambda t} + (x_2)_p \tag{44}$$

where $(x_2)_p$ represents the particular solution for equation (43) and it is determined by the character and/or value of the sales forecast function Q(t). Equations (44) and (29) yield

$$\overline{\theta}(t) = A_1e^{\lambda t} + A_2e^{-\lambda t} + (x_2)_p \tag{45}$$

where

$$\lambda = \sqrt{\frac{C_A}{C_B}} . \tag{46}$$

A_1 and A_2 are constants which may be determined from the initial conditions associated with the problem.

A MORE COMPREHENSIVE MODEL

We shall now study a model which, as in the original case, assumes desired preference levels for the production rate and inventory volumes. In addition, this model takes into account the costs associated with any changes in the production rate.

We shall assume that, in general, the rate of change in finished-goods inventories is given by

$$\frac{dI(t)}{dt} = P(t) - Q(t) \quad , \quad I(0) = I_o, \tag{47}$$

where I(t), P(t) and Q(t) represent the inventories, production and sales forecast for time t respectively.

We assume also that the rate of costs from holding inventories and/or stockouts will be approximated by the quadratic

$$C_I[I(t) - \bar{I}]^2 \tag{48}$$

where $\bar{I}$ is the desired inventory level. The rate at which manufacturing costs due to deviations from the desired plant operation level, $\bar{P}$, can be approximated by the quadratic*

$$C_p[P(t) - \bar{P}]^2 \tag{49}$$

and the rate of costs associated with changes in the production rate is approximated by

$$C_R[\frac{dP(t)}{dt}]^2. \tag{50}$$

Therefore, the total cost for the planning period T is given by

$$CT = \int_o^T \{C_I[I(t) - I]^2 + C_p[P(t) - \bar{P}]^2 + C_R[\frac{dP(t)}{dt}]^2\}dt \tag{51}$$

where C_I, C_P, and C_R are constants.

Here again the problem is that of determining the optimum production rate so that the sum of costs for the planning period as given by equation (51) is minimized. The objective function to be minimized can then be written

$$S = \int_o^T \{C_I[I(t) - \bar{I}]^2 + C_p[P(t) - \bar{P}]^2 + C_R[\frac{dP(t)}{dt}]^2\}dt. \tag{52}$$

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*$\bar{I}$ and $\bar{P}$ are both assumed to be constant for simplicity.

OPTIMIZATION OF THE MODEL

To apply the maximum principle, let us define

$$\theta(t) = P(t) \quad , \quad \theta(0) = P_o , \tag{53}$$

and

$$x_1(t) = I(t). \tag{54}$$

Then, equation (47) becomes

$$\frac{dx_1}{dt} = \theta(t) - Q(t) \quad , \quad x_1(0) = I_o, \tag{55}$$

We introduce $x_2(t)$ such that

$$x_2(t) = \int_o^t \{C_I[x_1 - \bar{I}]^2 + C_p[\theta - \bar{P}]^2 + C_R[\frac{d\theta}{dt}]^2\}dt, \tag{56}$$

or

$$\frac{dx_2}{dt} = C_I[x_1 - \bar{I}]^2 + C_p[\theta - \bar{P}]^2 + C_R[\frac{d\theta}{dt}]^2, \; x_2(0) = 0. \tag{57}$$

Since the system defined by equation (55) and (57) is not given in the standard form required by the maximum principle*, we must first convert the system to the required standard form before the Hamiltonian function can be formulated. To perform this transformation we introduce

$$x_3(t) = \theta(t) \tag{58}$$

and a new decision variable

$$\omega = \frac{d\theta}{dt}. \tag{59}$$

Therefore,

$$\frac{dx_3}{dt} = \omega \quad , \quad x_3(0) = \theta(0) = P_o. \tag{60}$$

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*This system belongs to that type of problem containing memory in the decision.

In terms of equations (58) through (60), equations (55) and (57) can be written

$$\frac{dx_1}{dt} = x_3 - Q \quad , \quad x_1(0) = I_o, \tag{61}$$

$$\frac{dx_2}{dt} = C_I[x_1 - \bar{I}]^2 + C_p[x_3 - \bar{P}]^2 + C_R[\omega]^2, \; x_2(0) = 0. \tag{62}$$

The enlarged system defined by equations (60) through (62) is now in the standard form. The objective function becomes

$$S = \sum_{i=1}^{3} c_i x_i(T) = x_2(T) \tag{63}$$

from which we obtain $c_1 = c_3 = 0$, $c_2 = 1$.

The Hamiltonian function can now be written

$$H(x_1 z_1 \omega) = z_1\{x_3 - Q\} + z_2\{C_I[x_1 - \bar{I}]^2 + C_p[x_3 - \bar{P}]^2 + C_R[\omega]^2\} + z_3\{\omega\} . \tag{64}$$

From equation (64) we obtain the adjoint variables as

$$\frac{dz_1}{dt} = - \frac{\partial H}{\partial x_1} = -2z_2[x_1 - \bar{I}]C_I, \quad z_1(T) = c_1 = 0, \tag{65}$$

$$\frac{dz_2}{dt} = - \frac{\partial H}{\partial x_2} = 0 \quad , \quad z_2(T) = c_2 = 1, \tag{66}$$

and

$$\frac{dz_3}{dt} = - \frac{\partial H}{\partial x_3} = -z_1 - 2z_2[x_3 - \bar{P}]C_p, \quad z_3(T) = c_3 = 0. \tag{67}$$

Solving equation (66) for $z_2(t)$ we obtain

$$z_2(t) = 1, \qquad 0 \le t \le T. \tag{68}$$

Hence, the Hamiltonian function becomes

$$H = z_1\{x_3 - Q\} + C_I[x_1 - \bar{I}]^2 + C_p[x_3 - \bar{P}]^2 + C_R[\omega]^2 + z_3\{\omega\}. \tag{69}$$

Applying the optimality condition $\frac{\partial H}{\partial \omega} = 0$ to equation (69), we obtain

$$\frac{\partial H}{\partial \omega} = 0 = 2C_R\omega + z_3 \tag{70}$$

or

$$z_3(t) = -2C_R\omega. \tag{71}$$

Differentiating equation (67) with respect to time and substituting equations (65) and (68) into the resulting equation yields

$$\frac{d^2z_3}{dt^2} = 2C_I[x_1 - \bar{I}] - 2C_p \frac{dx_3}{dt} . \tag{72}$$

The differentiation of equation (72) and the substitution of equation (61) into the result gives

$$\frac{d^3z_3}{dt^3} = 2C_I[x_3 - Q(t)] - 2C_p \frac{d^2x_3}{dt^2} . \tag{73}$$

Let us substitute equation (60) into equation (71). We obtain

$$z_3 = -2C_R \frac{dx_3}{dt} . \tag{74}$$

Differentiating equation (74) three times with respect to time and substituting $z_3(t)$ back into equation (73) gives

$$\frac{d^4x_3}{dt^4} - \frac{C_p}{C_R} \frac{d^2x_3}{dt^2} + \frac{C_I}{C_R} x_3 = \frac{C_I}{C_R} Q(t). \tag{75}$$

Using the identity defined by equation (58), equation (75) becomes

$$\frac{d^4\theta}{dt^4} - \frac{C_p}{C_R} \frac{d^2\theta}{dt^2} + \frac{C_I}{C_R} \theta = \frac{C_I}{C_R} Q(t). \tag{76}$$

The solution of the differential equation (76) gives, by virtue of equation (70), the optimum solution for the decision variable (production rate), $\bar{\theta}(t)$. In terms of the differential

operator

$$D = \frac{d}{dt} \tag{77}$$

equation (76) can be written

$$\left[D^4 - \frac{C_p}{C_R} D^2 + \frac{C_I}{C_R}\right] \theta = \frac{C_I}{C_R} Q(t) \tag{78}$$

Let us define

$$(\theta)_p \tag{79}$$

to be the particular solution of equation (78). It is determined once the sales forecast function, Q(t), is defined. The complementary part of the solution to equation (78) can be obtained by letting

$$D^4 - \frac{C_p}{C_R} D^2 + \frac{C_I}{C_R} \quad = 0 \tag{80}$$

and solving for the roots of this polynomial. We obtain

$$\lambda_1 = + \sqrt{a + b}$$

$$\lambda_2 = + \sqrt{a - b}$$

$$\lambda_3 = - \sqrt{a + b}$$

$$\lambda_4 = - \sqrt{a - b} \tag{81}$$

where

$$a = \frac{C_p}{2C_R} \tag{82}$$

$$b = \sqrt{\left(\frac{C_p}{2C_R}\right)^2 - \frac{C_I}{C_R}} \,. \tag{83}$$

The form of the complementary solution for $\overline{\theta}(t)$ is obviously determined by the character of the roots defined in equation (81). These roots are, of course, ultimately determined by the values

of the parameters of the model, C_I, C_P, and C_R. Since these parameters are always positive and real quantities, we may distinguish three main feasible forms of solution for $\overline{\theta}(t)$:

1. If

$$\left(\frac{C_p}{2C_R}\right)^2 > \frac{C_I}{C_R} \quad \text{or} \quad C_p^{\,2} > 4C_RC_I, \tag{84}$$

the roots given by equation (81) are all real and distinct. Under this case, the optimum decision (optimum production rate) is given by

$$\overline{\theta}(t) = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t} + A_3e^{\lambda_3 t} + A_4e^{\lambda_4 t} + (\theta)_p \tag{85}$$

where A_1, A_2, A_3, and A_4 are constants to be determined from the boundary condition associated with the problem.

2. If

$$\left(\frac{C_p}{2C_R}\right)^2 = \frac{C_I}{C_R} \quad \text{or} \quad C_p^{\,2} = 4C_RC_I, \tag{86}$$

then b = 0, and the roots in equation (81) become two distinct pairs of real roots. That is

$$\lambda_1 = \lambda_3 = +\sqrt{a} \ ,$$

$$\lambda_2 = \lambda_4 = -\sqrt{a} \ . \tag{87}$$

Under this case, the optimum decision becomes

$$\overline{\theta}(t) = A_1e^{\lambda t} + A_2te^{\lambda t} + A_3e^{-\lambda t} + A_4te^{-\lambda t} + (\theta)_p$$

$$= e^{\lambda t}(A_1 + A_2t) + e^{-\lambda t}(A_3 + A_4t) + (\theta)_p \tag{88}$$

where

$$\lambda = +\sqrt{\frac{C_p}{2C_R}} \ . \tag{89}$$

3. If

$$\left(\frac{C_p}{2C_R}\right)^2 < \frac{C_I}{C_R}\ ,\ \text{or}\ C_p^{\ 2} < 4C_IC_R \tag{90}$$

b becomes an imaginary quantity and the four roots in equation (81) take the form

$$\begin{aligned}\lambda_1 &= +\sqrt{a + \bar{b}i}\\ \lambda_2 &= +\sqrt{a - \bar{b}i}\\ \lambda_3 &= -\sqrt{a + \bar{b}i}\\ \lambda_4 &= -\sqrt{a - \bar{b}i}\end{aligned} \tag{91}$$

where $i = \sqrt{-1}$, and

$$\bar{b} = \sqrt{\frac{C_I}{C_R} - \left(\frac{C_p}{2C_R}\right)^2}\ . \tag{92}$$

The roots given by equation (91) can be easily transformed into pairs of conjugate mixed (real and imaginary) roots by using DeMoivre's theorem for roots of imaginary quantities. The nomenclature used in this theorem is graphically described in figure 1. If we define

$$a + \bar{b}i = r(\cos\rho + i\sin\rho) \tag{93}$$

where

$$r = \sqrt{a^2 + \bar{b}^2} \tag{94}$$

$$\cos\rho = \frac{a}{\sqrt{a^2 + \bar{b}^2}} \tag{95}$$

and

$$\sin\rho = \frac{\bar{b}}{\sqrt{a^2 + \bar{b}^2}} \tag{96}$$

and using the first root in the complex plane, the DeMoivre theorem can in our case be written

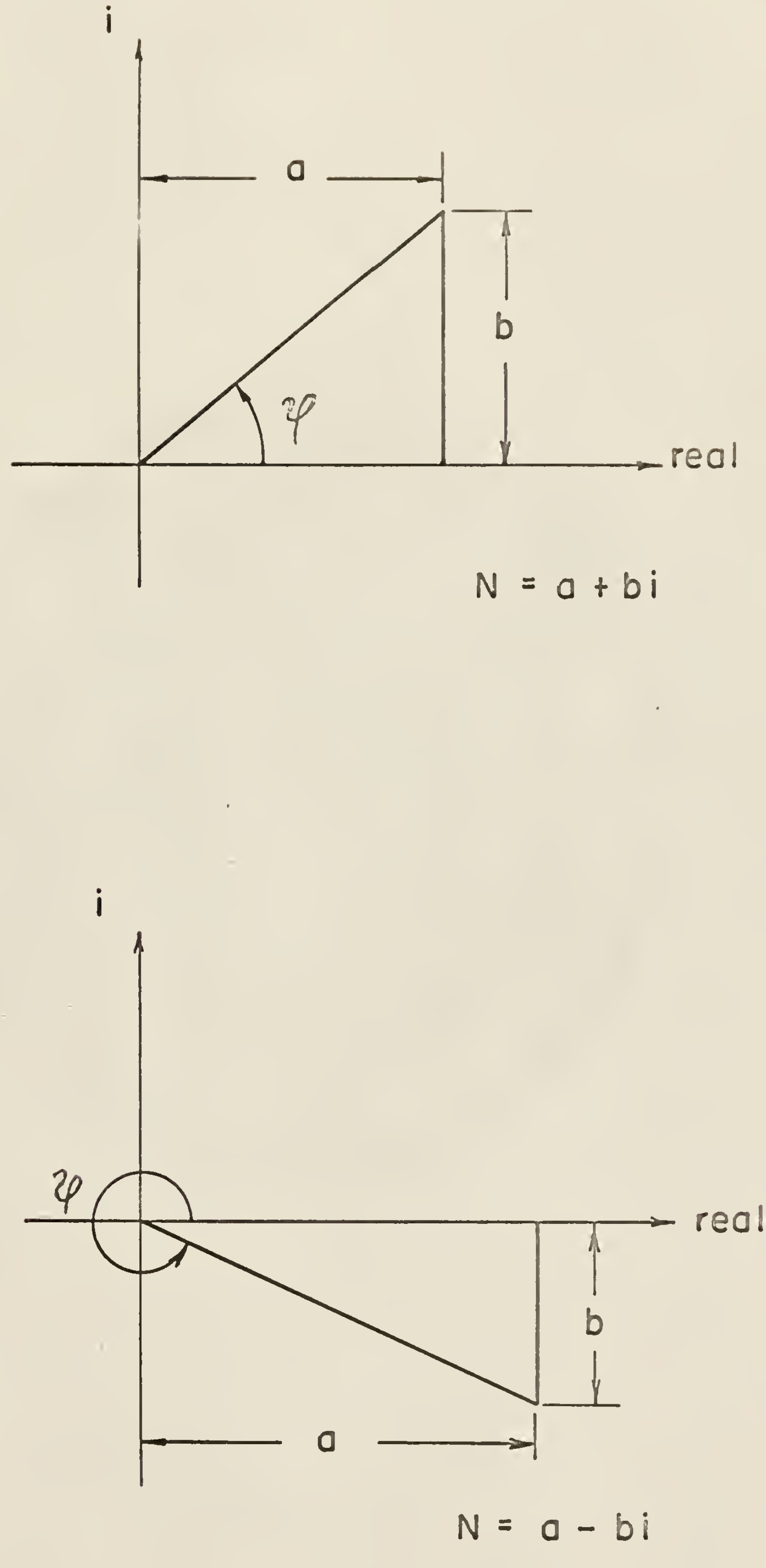

Fig. 1 Complex variable representation.

$$r\ (\cos\rho + i\sin\rho) \quad = \quad r\ (\cos\tfrac{\rho}{2} + i\sin\tfrac{\rho}{2}). \tag{97}$$

With the simultaneous use of equations (93) and (97) the following identities are obtained

$$\begin{aligned} \sqrt{a+\bar{b}i} &= \alpha + \beta i \\ \sqrt{a-\bar{b}i} &= -\alpha + \beta i \end{aligned} \tag{98}$$

where

$$\alpha = \sqrt{\frac{\sqrt{a^2+\bar{b}^2} + a}{2}} = \sqrt{\sqrt{\frac{C_I}{4C_R}} + \frac{C_p}{4C_R}} \tag{99}$$

and

$$\beta = \sqrt{\frac{\sqrt{a^2+\bar{b}^2} - a}{2}} = \sqrt{\sqrt{\frac{C_I}{4C_R}} - \frac{C_p}{4C_R}} \tag{100}$$

By virtue of the condition given in equation (90), it can be proved that β is always a real quantity.

In terms of the identities given by equation (98), the four roots of equation (91) become

$$\begin{aligned} \lambda_1 &= \sqrt{a+\bar{b}i} &&= \alpha + \beta i \\ \lambda_2 &= \sqrt{a-\bar{b}i} &&= -\alpha + \beta i \\ \lambda_3 &= -\sqrt{a+\bar{b}i} &&= -\alpha - \beta i \\ \lambda_4 &= -\sqrt{a-\bar{b}i} &&= \alpha - \beta i. \end{aligned} \tag{101}$$

It is now apparent that λ_1 and λ_4, as well as λ_2 and λ_3 constitute two pairs of conjugated complex roots.

The optimum decision policy under this case, therefore,

is given by the function

$$\bar{\theta}(t) = e^{\alpha t}[A_1 \cos \beta t + A_2 \sin \beta t] +$$

$$e^{-\alpha t}[A_3 \cos \beta t + A_4 \sin \beta t]. \tag{102}$$

Under any one of the three conditions discussed above, the knowledge of a particular sales forecast function Q(t) is required in order to obtain the total solution for the optimum decision function as given either by equation (85), (88) or (102). Once the optimum decision function, $\bar{\theta}(t)$, has been determined, the actual inventory level resulting from this decision can be obtained directly from equation (61) using a direct integration. That is

$$x_1(t) = \int_0^t [\bar{\theta}(t) - Q(t)]dt + K \tag{103}$$

where K is an integration constant to be determined from the initial conditions, that is, the inventory level existing at the beginning of the planning period.

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6. MATHEMATICAL MODELS FOR THE OPTIMIZATION OF EQUIPMENT INVESTMENT

INTRODUCTION

A problem faced by a manufacturing company when investing in production equipment is that of maximizing the total net worth of such an investment. The sales of goods generate a continuous stream of revenue over the productive life of the equipment. Associated in time with this stream of revenue is a corresponding stream of expenses necessary for the production of these goods. The difference between these two streams represents the return on investment before deducting capital costs.

In this paper, a basic model for profit maximization treated by Preinreich (5) and others (1, 6) is introduced and then a more comprehensive model is presented. The applicability of the maximum principle (2, 4) in optimizing equipment investment is then demonstrated by using both models.

A CLASSICAL MODEL FOR PROFIT MAXIMIZATION

From the efficiency point of view, two general kinds of equipment may be distinguished: the "constant efficiency" and the "diminishing efficiency" types. Under the first category we may classify those items whose efficiency remains fairly constant throughout their service lives and whose service terminates abruptly with their first failure. An electric light bulb is the best example of this type of equipment. To the second classification belong those durable goods whose service life may be extended almost indefinitely if their component parts are replaced

or repaired as necessary. This type of equipment is characterized by a decline in productivity or an increase in maintenance costs as they are used over time.

The economics of replacement associated with these two types of equipment are quite different. For those goods displaying a constant efficiency, a probability distribution for the length of their lives may be obtained from life tests and various replacement policies may be evaluated on the basis of this distribution. Since there is no cost of declining efficiency associated with the problem, the analysis is very often reduced to a comparison of the expected values of the several alternatives.

If a simple piece of equipment of the diminishing efficiency type earns revenue according to some function, $R(t)$, and incurs a stream of maintenance and operating expenses given by the function $U(t)$, then the net present value of the investment to the firm is given by (5)

$$V_1 = \int_0^T [R(t) - U(t)]e^{-it}\,dt + D(T)e^{-iT} - B, \tag{1}$$

where

V_1 = net present worth of the investment,

B = installed cost of the equipment,

T = economic life of the equipment,

$D(T)$ = salvage value of the equipment at time T,

i = annual rate of interest.

Note that the expense function, $U(t)$, excludes depreciation costs and interest on investment in order to avoid double counting these

items in equation (1).

For an infinite chain of similar machines, the present worth formula given by equation (1) becomes (5)

$$V_\infty = \{\int_o^T [R(t) - U(t)]e^{-it}\, dt + D(T)e^{-it} - B\}\frac{1}{(1-e^{-iT})}. \qquad (2)$$

Equations (1) and (2) are very often of the discrete character in which a summation of the discrete revenue and expenditures discounted to the present replaces the integrals of equations (1) and (2).

We shall consider only the continuous case for a single machine. The objective function for the case under consideration can be written

$$S = V_1. \qquad (3)$$

The problem, therefore, becomes that of determining the optimum life of the equipment, $\overline{T}$, so that the net present value as given by equation (1) attains its maximum.

Optimization based on the simple model

Before we proceed to solve the optimization problem stated above, let us briefly discuss the applicability of the maximum principle to the problem.

Figure 1 is a graphical representation of the optimum trajectory concept used in such variational techniques as the maximum principle and the classical calculus of variations. The problem usually treated by these techniques is that of selecting a decision function, $\overline{\theta}(t)$, to obtain an optimum trajectory, $\overline{x}(t)$, which renders the objective function, $S(t)$, an extremum in the

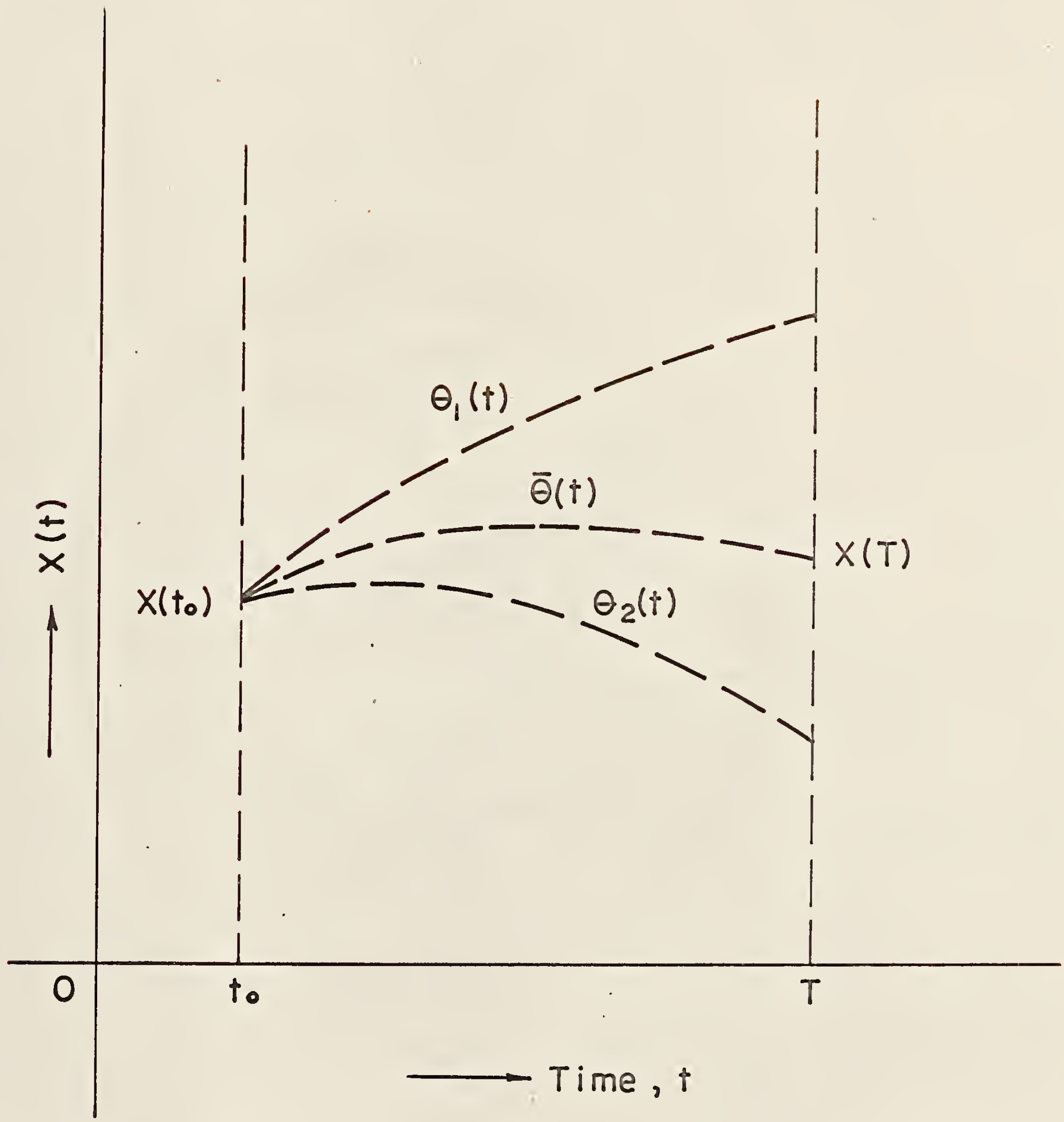

Fig. I Optimum Trajectory with the decision vector, $\Theta(t)$ as the parameter.

closed interval, $t_o \leq t \leq T$. Very often the boundaries of the interval are also to be chosen. These techniques are also applicable when the initial and/or final conditions are specified.

For the optimization under consideration, the determination of the optimum upper bound, $\overline{T}$, alone will extremize the objective function. That is, the problem belongs to the "zero control" category in which no decision function is involved and, consequently, there are no trajectories involved. This problem, therefore, does not belong to a class of problems in which the application of variational techniques is advantageous. This type of problem is amenable to solution by the classical calculus.

Taking the derivative of equation (1) with respect to T and applying the condition

$$\frac{dV}{dT} = 0 \tag{4}$$

given by the classical calculus, we obtain

$$R(T) - U(T) = iD(T) - D'(T). \tag{5}$$

If the functions for revenue, expenditure and depreciation are known, the optimum service life, $\overline{T}$, can be obtained from equation (5) be means of a simple numerical analysis.

<u>Solution by the maximum principle</u>

In order to apply the maximum principle, let us define

$$x_1(t) = \int_o^t [R(t) - U(t)]e^{-it}\, dt, \tag{6}$$

$$\frac{dx_1}{dt} = [R(t) - U(t)]e^{-it}, \quad x_1(o) = 0\ , \tag{7}$$

$$x_2(t) = D(t)e^{-it} - B\ , \tag{8}$$

$$\frac{dx_2}{dt} = D'(t)e^{-it} - iD(t)e^{-it},\ x_2(o) = 0\ , \tag{9}$$

where $D'(t) = dD/dt$.

Since the system defined by equations (7) and (9) is non-autonomous (the right hand sides of equations (7) and (9) depend explicitly on time), we shall introduce a new state variable, x_3, defined by

$$\frac{dx_3}{dt} = 1,\quad x_3(o) = t_0 = 0\ . \tag{10}$$

It is obvious that $x_3 = t$.

The objective function as given by equation (3) can now be written

$$S = \sum_{i=1}^{3} c_i\ x_i(T)$$

$$= x_1(T) + x_2(T), \tag{11}$$

therefore, $c_1 = c_2 = 1$, $c_3 = 0$.

The Hamiltonian function and the adjoint variables are (2,4)

$$H = z_1 \frac{dx_1}{dt} + z_2 \frac{dx_2}{dt} + z_3 \frac{dx_3}{dt}$$

$$= z_1\{[R(t) - U(t)]e^{-ix_3}\} +$$

$$+ z_2\ \{D'(t)e^{-ix_3} - iD(t)e^{-ix_3}\} + z_3\{1\}, \tag{12}$$

$$\frac{dz_1}{dt} = - \frac{\partial H}{\partial x_1} = 0, \tag{13}$$

$$z_1(T) = c_1 = 1, \tag{14}$$

$$\frac{dz_2}{dt} = - \frac{\partial H}{\partial x_2} = 0, \tag{15}$$

$$z_2(T) = c_2 = 1, \tag{16}$$

$$\frac{dz_3}{dt} = -\frac{\partial H}{\partial x_3} = iz_1[R(t)-U(t)]e^{-ix_3}$$

$$+ iz_2 D'(t)e^{-ix_3} - i^2 z_2 D(t)e^{-ix_3}, \tag{17}$$

$$z_3(T) = c_3 = 0. \tag{18}$$

Solving equations (13) through (16) we obtain

$$z_1(t) = 1, \ 0 \leq t \leq T, \tag{19}$$

$$z_2(t) = 1, \ 0 \leq t \leq T . \tag{20}$$

Equations (17) and (18) can now be solved for $z_3(t)$ to yield

$$z_3(t) = -i \int_t^T [R(t)-U(t)-iD(t) + D'(t)]e^{-it}dt . \tag{21}$$

Substituting equations (19), (20) and (21) back into equation (12), the Hamiltonian function becomes

$$H = [R(t)-U(t)]e^{-it} + D'(t)e^{-it} - iD(t)e^{-it}$$

$$- i \int_t^T [R(t)-U(t)-iD(t)+iD(t)+D'(t)]e^{-it} dt . \tag{22}$$

According to the maximum principle, the optimal decision function $\overline{\theta}(t)$ which makes S maximum makes H maximum and fixed at zero, for time T not fixed, that is (2, 4)

$$\max H = 0 \quad , \quad t_0 \leq t \leq T.$$

Using this optimality condition and substituting $t = T$ into equation (22) finally we obtain

$$R(T) - U(T) = iD(T) - D'(T). \tag{23}$$

Equation (23) is the same solution given by the classical differential calculus. It can be seen that the calculus solution

requires only one differentiation while the maximum principle requires considerably more manipulation than that required in the use of the calculus.

A numerical example

In order to illustrate, let us assume that the total cost of installation, B, for a given piece of equipment is \$10,000 and that this machine will generate a revenue function of the form (1, 6)

$$R(t) = 6{,}000\ (1 - 0.02t). \tag{24}$$

The annual rate of expenses has been estimated to be \$2,000 for the first year and it is expected to increase at a rate of 15% per year due to additional maintenance and service required to keep the machine in operation. Therefore,

$$U(t) = 2{,}000\ (1+0.15t). \tag{25}$$

These estimates are based on the company's experience with similar machines in the past. It has been the company's policy to assume an exponential depreciation for this type of machine of the form

$$\begin{aligned} D(t) &= Be^{-kt} \\ &= 10{,}000\ e^{-0.20t}. \end{aligned} \tag{26}$$

All alternative proposals are evaluated using an annual rate of interest of 10%. On the basis of these figures, we want to know how long the machine should be kept in operation in order to maximize any profit derived from the investment over and above the prescribed rate of interest.

From equation (26) we obtain

$$D'(t) = -\ 2{,}000\ e^{-0.20t} \tag{27}$$

and substituting equations (24) through (26) into equation (23) gives

$$400 - 42t = 300\ e^{-0.20t}. \qquad (28)$$

A solution of equation (28) for $\overline{T}$ gives $\overline{T}$ = 7.98 years. For this investment time, the present value of V_1 = \$6,021 is obtained from equation (1). This is an optimum.

A MORE REALISTIC MODEL

We assumed in the model discussed above that the investment time, T, is solely responsible for the maximization of profits. It is easy to visualize, however, that under actual conditions there are other factors which are equally or more significant than the investment time and which should therefore be brought into the analysis. One such factor is the production rate at which the equipment is operated. In the analysis that follows, the production rate is introduced as the second decision variable which is dependent on time.

The manner in which the production rate affects the operation of the system varies with the market conditions (revenue function), the manufacturing process (expense function) and the type of equipment used (depreciation function). These factors are not completely independent of each other but for computational purposes they may be considered so without lessening the efficiency of the model.

A mathematical model which accounts for all possible forms of variation in the system is obviously unattainable and therefore simplifying assumptions are made here.

1. The company's share of the market, M_s, remains constant

throughout the investment time, T.

2. The cost of any shortage is negligible* and no inventory is carried. Consequently, we can write

$$0 \leq P(t) \leq M_s, \qquad 0 \leq t \leq T, \tag{29}$$

where P(t) is the production rate.

3. The amount of maintenance and servicing required per unit time, M(P,t), is proportional to the cumulative service obtained from the machine up to time t, $\int_0^t P(t)dt$, and is inversely proportional to the total expected service of the machine, A. We may write

$$M(P,t) = m\left[\frac{1}{A}\int_0^t P(t)dt\right]^{\gamma} E \tag{30}$$

where E is the fixed overhead cost associated with the machine ($/time). The constants m and γ are positive parameters characteristic of each type of machine and can be determined from the company record (or manufacturer's data) on similar machines in the past.

It can be derived from equation (30) that when the expected production has been obtained from the machine,

$$\int_0^t P(t)dt = A \text{ (units produced)},$$

the rate of maintenance and servicing required becomes

$$M(P,t) = m\,E \text{ (\$/time)}.$$

Figure 2 is a graphical representation of the effect of the value of γ on the maintenance cost function. Both m and γ must be chosen according to the maintenance conditions dictated by each

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*It will be seen later that, despite of this assumption, the conditions for optimality require a rate of production as close as possible to the market share.

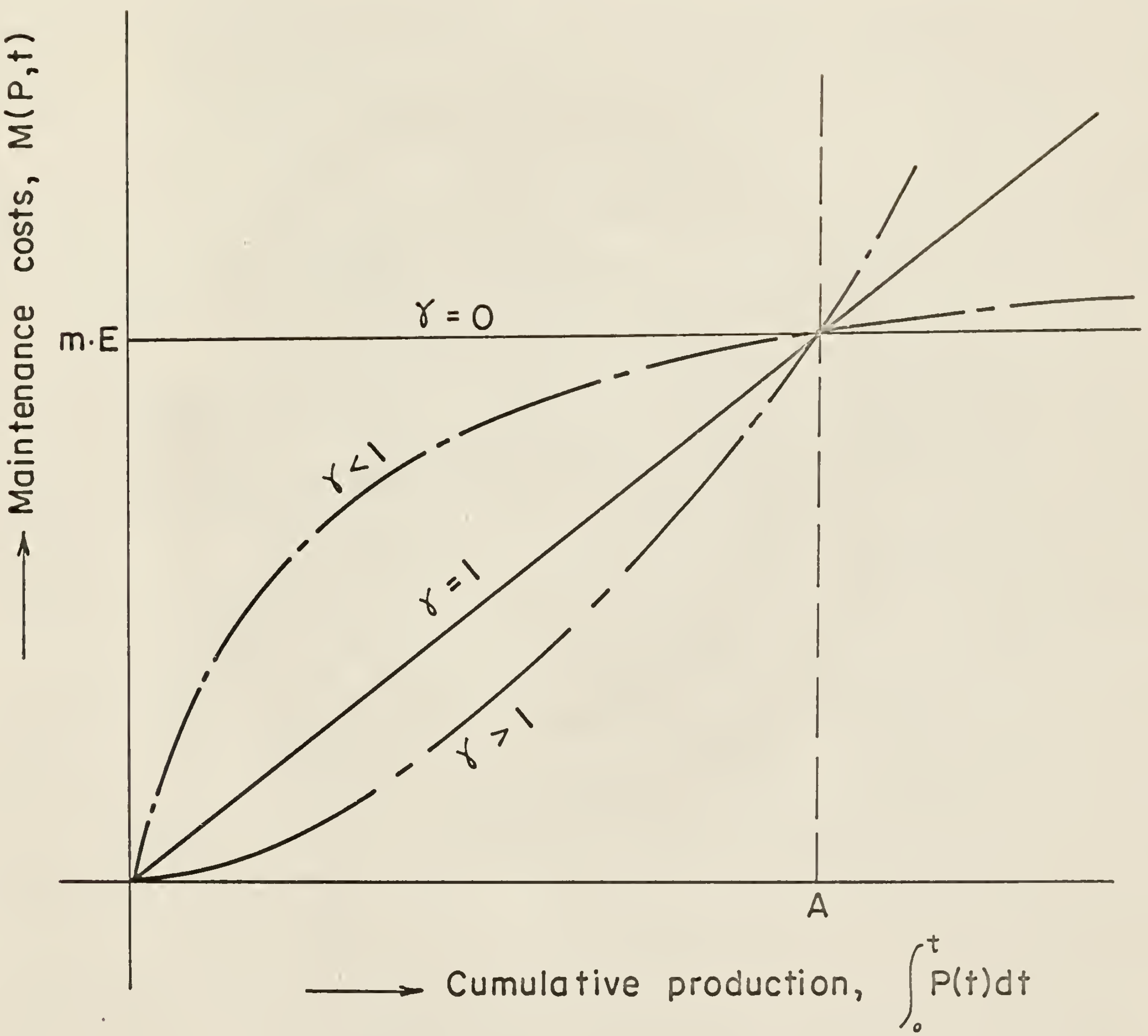

Fig. 2 The effect of γ on the maintenance cost function.

particular type of machine. In all cases, these parameters as well as all other parameters in the model may be functions of time but, for simplicity, we shall treat them as constants throughout the analysis.

4. The revenue function is proportional to the production rate since we assume a constant sale price, S_P. Then,

$$R(P,t) = S_P\ P(t). \tag{31}$$

Similarly, the function

$$VC(P,t) = C_V\ P(t) \tag{32}$$

represents all variable costs with C_V being the per-unit variable cost.

5. With the total installed cost, B, and a constant rate of depreciation, k, the salvage value of the machine at time t is given by

$$D(t) = B\ e^{-kt}. \tag{33}$$

Using the net present worth as the criteria for optimality we write

$$V = \int_o^T [R(P,t)-VC(P,t)-E-M(P,t)]e^{-it}dt + D(t)e^{-iT} - B. \tag{34}$$

The term under the integral sign represents the present worth of revenues minus all expenses except depreciation. The two terms outside the integral sign may be understood as the net total cost of buying the equipment and selling it at a price D(T) after T years of use.

Let us, for simplicity, assume $\gamma=2$. Substituting equations (29) through (33) into equation (34) and rearranging terms we obtain

$$V = \int_{o}^{T} \{(S_p - C_v)P(t) - E[1 + m(\frac{1}{A}\int_{o}^{t} P(\tau)d\tau)^2]\}e^{-it}dt + B[e^{-(k+i)T} - 1]. \tag{35}$$

Our objective is to maximize the net present value of the investment as given in equation (35) by choosing the most profitable rate of production, $\overline{P}(t)$, during the optimum investment time, $\overline{T}$. We shall accomplish this through the use of the maximum principle.

<u>Optimization based on the more realistic model</u>

To apply the maximum principle let the production rate be the decision variable, i.e.,

$$\theta(t) = P(t), \qquad o \leq \theta(t) \leq \theta_{max} . \tag{36}$$

The state variables are defined as follows:

$$x_1(t) = \frac{1}{A}\int_{o}^{t} \theta(\tau)d\tau , \tag{37}$$

$$\frac{dx_1}{dt} = \frac{\theta(t)}{A} , \qquad x_1(o) = 0, \tag{38}$$

$$x_2(t) = B[e^{-(k+i)t} - 1], \tag{39}$$

$$\frac{dx_2}{dt} = -(k+i)\ Be^{-(k+i)t}, \ x_2(o) = 0, \tag{40}$$

$$x_3(t) = \int_{o}^{t} [q\theta(t) - E(1 + mx_1^2)]\ e^{-it}dt, \tag{41}$$

$$\frac{dx_3}{dt} = [q\theta(t) - E(1+mx_1^2)]e^{-it}, \ x_3(o) = 0, \tag{42}$$

where

$$q = (S_p - C_v) > 0 . \tag{43}$$

q is the unit logistic margin (3), that is, the sale price minus the variable cost per unit.

Since the system defined by equations (38), (40) and (42) is nonautonomous (the right hand sides depend explicitly on time), we shall introduce an additional state variable, x_4, defined by

$$x_4(t) = t,$$

$$\frac{dx_4}{dt} = 1, \qquad x_4(0) = t_0 = 0. \tag{44}$$

The objective function to be maximized now becomes

$$S = \sum_{i=1}^{4} c_i x_i(T) = x_2(T) + x_3(T) \ . \tag{45}$$

Therefore,

$$c_1 = c_4 = 0, \qquad c_2 = c_3 = 1.$$

The Hamiltonian function and adjoint variables can be written as (2, 4)

$$H = z_1 \{\frac{\theta}{A}\} + z_2 \{- (k+i)B\ e^{-(k+i)x_4}\}$$

$$+ z_3 \{q\theta - E(1+m\ x_1^2)\}e^{-ix_4} + z_4\{1\}, \tag{46}$$

$$\frac{dz_1}{dt} = - \frac{\partial H}{\partial x_1} = 2z_3 E m x_1 e^{-ix_4}, \ z_1(T) = c_1 = 0, \tag{47}$$

$$\frac{dz_2}{dt} = - \frac{\partial H}{\partial x_2} = 0, \qquad z_2(T) = c_2 = 1, \tag{48}$$

$$\frac{dz_3}{dt} = - \frac{\partial H}{\partial x_3} = 0, \qquad z_3(T) = c_3 = 1, \tag{49}$$

$$\frac{dz_4}{dt} = - \frac{\partial H}{\partial x_4} = -z_2(k+i)^2\ Be^{-(k+i)x_4} +$$

$$z_3 i[q\theta - E(1+mx_1^2)]e^{-ix_4}\ , \quad z_4(T) = c_4 = 0. \tag{50}$$

Solving equations (48) and (49) we obtain

$$z_2(t) = 1, \qquad 0 \leq t \leq T \tag{51}$$

$$z_3(t) = 1, \qquad 0 \leq t \leq T\ . \tag{52}$$

Substituting equations (51) and (52) into equation (46) and separating terms we obtain

$$H = H^* - (k+i)Be^{-(k+i)x_4} - E(1+m\ x_1^2)e^{-ix_4} + z_4 \tag{53}$$

where

$$H^* = (\frac{z_1}{A} + q)\ \theta(t) \tag{54}$$

is the variable part, with respect to $\theta(t)$, of the Hamiltonian.

It is now apparent from equation (54) that the optimal control associated with this problem is of the "on - off" or "Bang - Bang" type in which the variable part of the Hamiltonian function takes the form (2)

$$H^* = h\ \theta. \tag{55}$$

This type of control is characterized by the variation of the decision variable, θ, which may take its maximum value (when h is positive) or its minimum value (when h is negative) in order to maximize the Hamiltonian function (2).

Let h be the coefficient of θ in equation (54), that is

$$h = \frac{z_1}{A} + q\ . \tag{56}$$

Then the optimal control which renders the Hamiltonian its maximum value will be

$$\bar{\theta} = \begin{cases} \theta_{max} & \text{(Production at the maximum rate) if } h > 0 \\ 0 & \text{(no production at all) if } h < 0 \end{cases} \tag{57}$$

where $\overline{\theta}$ is the optimal decision policy (optimum production rate) which will maximize the objective function. Recall that $\theta > 0$. We shall now find the switching time, t_s, at which h changes sign. The switching time may be found from the condition

$$h(t_s) = 0. \tag{58}$$

From the optimality condition obtained in equation (57) it is seen that θ is not a continuous function of time and that it may take only one of the extreme values. For computational purposes, then, θ may be assumed to be a constant,

$$\theta = \rho \qquad \text{where } \rho = \begin{cases} \theta_{max}, \\ 0 \quad . \end{cases} \tag{59}$$

Using equation (59) and solving for x_1 and z_1 in equations (38) and (47) with the boundary conditions, $x_1(0) = 0$, and $z_1(T) = 0$, we obtain

$$x_1(t) = \frac{\rho t}{A} , \tag{60}$$

$$z_1(t) = \frac{-2mE\rho}{Ai^2} [(iT+1)e^{-iT} \quad -(it+1)e^{-it}] , \tag{61}$$

and h can now be written

$$h = - \frac{2mE\rho}{A^2 i^2} [(iT+1)e^{-iT} -(it+1)e^{-it}] + q . \tag{62}$$

Since $q > 0$, it follows immediately from equation (62) that

$$h > 0 \qquad \text{for} \qquad 0 \leq t \leq T \tag{63}$$

and consequently,

$$t_s > T. \tag{64}$$

Since we are concerned only with the interval $0 \leq t \leq T$ at the end of which the service life of the machine is terminated, the

optimum production policy for this period is

$$\overline{P}(t) = \overline{\theta}(t) = \theta_{max} = \min \begin{cases} \text{Maximum Plant Capacity} \\ \\ M_s, \text{ the market share} \end{cases},$$

$$0 \leq t \leq T. \tag{65}$$

In order to maximize the total present worth of the investment, then, the maximum possible rate of production should be maintained throughout the service life of the machine. The rate of production, however, should not exceed the market share of the company since inventories are not allowed. The optimal condition given by equation (65) precludes the first part of assumption number two since the optimal condition minimizes shortages regardless of how inexpensive they may be. The assumption, however, is not redundant since the introduction of a shortage cost and its effect on the optimality condition were not tested.

It only remains to be determined what the optimum investment time $\overline{T}$ should be. According to the maximum principle, a condition for optimality is obtained by making use of the fact that max $H = 0$ for $t_0 \leq t \leq T$. Solving equation (50) for z_4, we obtain

$$z_4(t) = (k+i)\, B(e^{-(k+i)t} - e^{-(k+i)T}) + \frac{iq\theta - E}{i}(e^{-iT} - e^{-it})$$

$$+ \frac{mE\theta^2}{A^2 i^3}\left[(i^2t^2+2it+2)e^{-it} - (i^2T^2+2iT+2)e^{-iT}\right] . \tag{66}$$

Substituting z_1, z_2, z_3, z_4 and q into equation (46), the Hamiltonian function becomes

$$H = \frac{2Em\theta^2}{A^2 i^2}\left[e^{-it}(it+1) - e^{-iT}(iT+1)\right] - (k+i)Be^{-(k+i)T}$$

$$+ \{q\theta - E[1 + m(\frac{\theta t}{A})^2]\}e^{-it} + \frac{E - iq\theta}{i}\,(e^{-it} - e^{-iT})$$

$$+ \frac{mE\theta^2}{A^2 i^3} [(i^2 t^2 + 2it + 2)e^{-it} - (i^2 T^2 + 2iT + 2)e^{-iT}] \quad . \tag{67}$$

Letting $t = \overline{T}$ and $H = 0$ in equation (67), we obtain

$$e^{-k\overline{T}} = \frac{(S_p - C_v)\ \overline{\theta} - E}{(k+i)\ B} - \frac{mE\overline{\theta}^2}{(k+i)A^2 B}\ \overline{T}^2 \ , \tag{68}$$

from which the optimum investment time $\overline{T}$ can be found.

Let us define

$$\alpha = \frac{(S_p - C_v)\ \overline{\theta} - E}{(k+i)B} \ , \tag{69}$$

$$\beta = \frac{mE\overline{\theta}^2}{(k+i)A^2 B} \ , \tag{70}$$

$$F_1 = e^{-k\overline{T}} \ , \tag{71}$$

$$F_2 = \alpha - \beta\ \overline{T}^2 \quad . \tag{72}$$

Equation (68), then, can be written as

$$F_1 = F_2 \ . \tag{73}$$

Note that the maximum values of F_1 and F_2 are 1 and α respectively, which occur at $\overline{T} = 0$ and both are monotonically decreasing functions of $\overline{T}$. As shown in Fig. 3, therefore, three situations must be considered in solving $\overline{T}$ from equation (73).

When $\alpha > 1$ only one real and positive root occurs at which the objective function (net present worth) attains a unique extremum.

When $\alpha_u \leq \alpha \leq 1$, there exist two positive real roots, which satisfy equation (73). α_u is the value of α at which the two roots coincide. In other words, when $\alpha = \alpha_u$, the curves

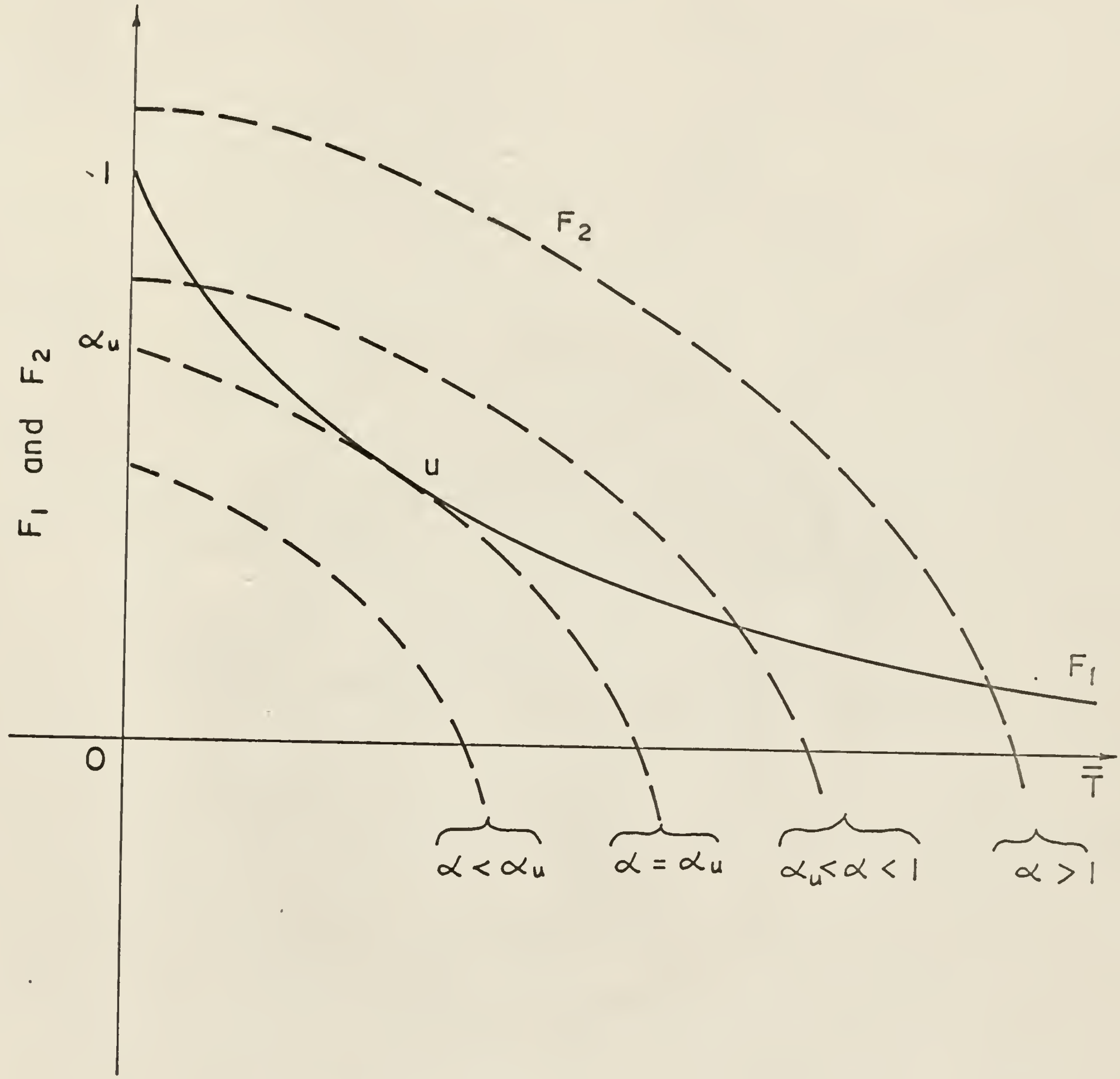

Fig. 3 Three situations in solving for $\bar{T}$.

representing F_1 and F_2 are tangential to each other.

When $\alpha < \alpha_u$, there is no real value solution to equation (73).

The tangential point of F_1 and F_2 where $\alpha = \alpha_u$ and $\bar{T} = \bar{T}_u$, can be determined by simultaneously solving equation (73) and the condition,

$$\left.\frac{dF_1}{d\bar{T}}\right|_{\bar{T} = \bar{T}_u} = \left.\frac{dF_2}{d\bar{T}}\right|_{\bar{T} = \bar{T}_u} . \tag{74}$$

Equations (73) and (74) can be written respectively as

$$e^{-k\bar{T}} = \alpha_u - \beta\,\bar{T}^2 \quad , \tag{75}$$

$$e^{-k\bar{T}_u} = 2\beta\bar{T}_u \quad , \tag{76}$$

and the solution for $\bar{T}_u$ can be carried out numerically. It can also be carried out approximately by representing the exponentials in equations (75) and (76) by the second order polynomial*

$$e^{-k\bar{T}_u} = 1 - k\bar{T}_u + \frac{k^2}{2}\,\bar{T}_u^2 \quad . \tag{77}$$

With this approximation equations (75) and (76) become

$$1 - k\bar{T}_u + \frac{k^2}{2}\,\bar{T}_u^2 = \alpha_u - \beta\bar{T}_u^2 \quad , \tag{78}$$

$$1 - k\bar{T}_u + \frac{k^2}{2}\,\bar{T}_u^2 = \frac{2\beta}{k}\,\bar{T}_u \quad . \tag{79}$$

From equation (79) we obtain

$$\bar{T}_u = \left(\frac{1}{k} + \frac{2\beta}{k^3}\right) \pm \sqrt{\left(\frac{1}{k} + \frac{2\beta}{k^3}\right)^2 - \frac{2}{k^2}} \quad . \tag{80}$$

---

* The error on α_u for $\beta = 0.05$ and $k = 0.25$, for instance, is approximately 4%.

Substituting equation (80) back into equation (78), α_u , is obtained as

$$\alpha_u = (1+\frac{2\beta}{k^2})^3 - (1+\frac{4\beta}{k^2}) - [(1+\frac{2\beta}{k^2})^2 - 1]\sqrt{(1+\frac{2\beta}{k^2})^2 - k^2} \quad . \tag{81}$$

Note that the negative sign in front of the radical in equation (80) is used in obtaining equation (81). The positive sign generates a value for α_u which is larger than one. Recall that $\alpha_u < 1$.

Once the optimum investment time is determined, the net present value of the investment can be calculated from equation (35). This gives

$$V_{max} = \frac{(S_p-C_v)\overline{\theta}-E}{i}(1-e^{-i\overline{T}}) + B[e^{-(k+i)\overline{T}} - 1]$$

$$+ \frac{Em\overline{\theta}^2}{A^2 i^3}[e^{-i\overline{T}}(i^2\overline{T}^2 + 2i\overline{T} + 2) - 2] \quad . \tag{82}$$

<u>Summary of results</u>

1. $\alpha > 1$. In this case equation (68) generates only one root at which a positive extremum is attained by the net present worth function.

2. $\alpha_u \le \alpha \le 1$. Two roots, $\overline{T}_1$ and $\overline{T}_2$, where $\overline{T}_2 > \overline{T}_1$, are obtained. $\overline{T}_1$ occurs before the break-even point indicating the time at which the maximum loss occurs. At $\overline{T}_2$ the net present worth of the investment is maximized.

3. $\alpha < \alpha_u$. Equation (68) fails to have a root and the net present worth function does not have an extremum. Losses increase indefinitely.

Figure 4 is a graphical representation of the behavior of the net present worth function under these three conditions.

As a closing statement, let us mention that, although the model is based on assumptions which could perhaps be considered too restrictive, we feel that the applicability of the maximum principle to this type of problems has been fully demonstrated. It is indeed very likely that the market share and/or the plant capacity may not remain constant as in the case of a growing market or a seasonal product. It can also be the case that many of the parameters of the model are time variables of one form or another. In all these cases, however, better reflection of actual conditions is feasible and the treatment of the model through the maximum principle differs from our case only in the handling of more complex functions.

A numerical example

A manufacturing company is contemplating the production of a well established item. A market survey has revealed the existence of a potential average demand for the product of 3,000 units per year and it is expected to remain at this level for several years in the future. The market price for this product is expected to remain at $5.00 per unit.

Three manufacturing methods are available to the company, all of which call for an investment of $10,000 to cover the cost of equipment and installation. The fixed overhead cost that would be allocated to this equipment has been estimated to be $2,000 per year. Regardless of which manufacturing method is used, the equipment is expected to produce 10,000 units before a complete

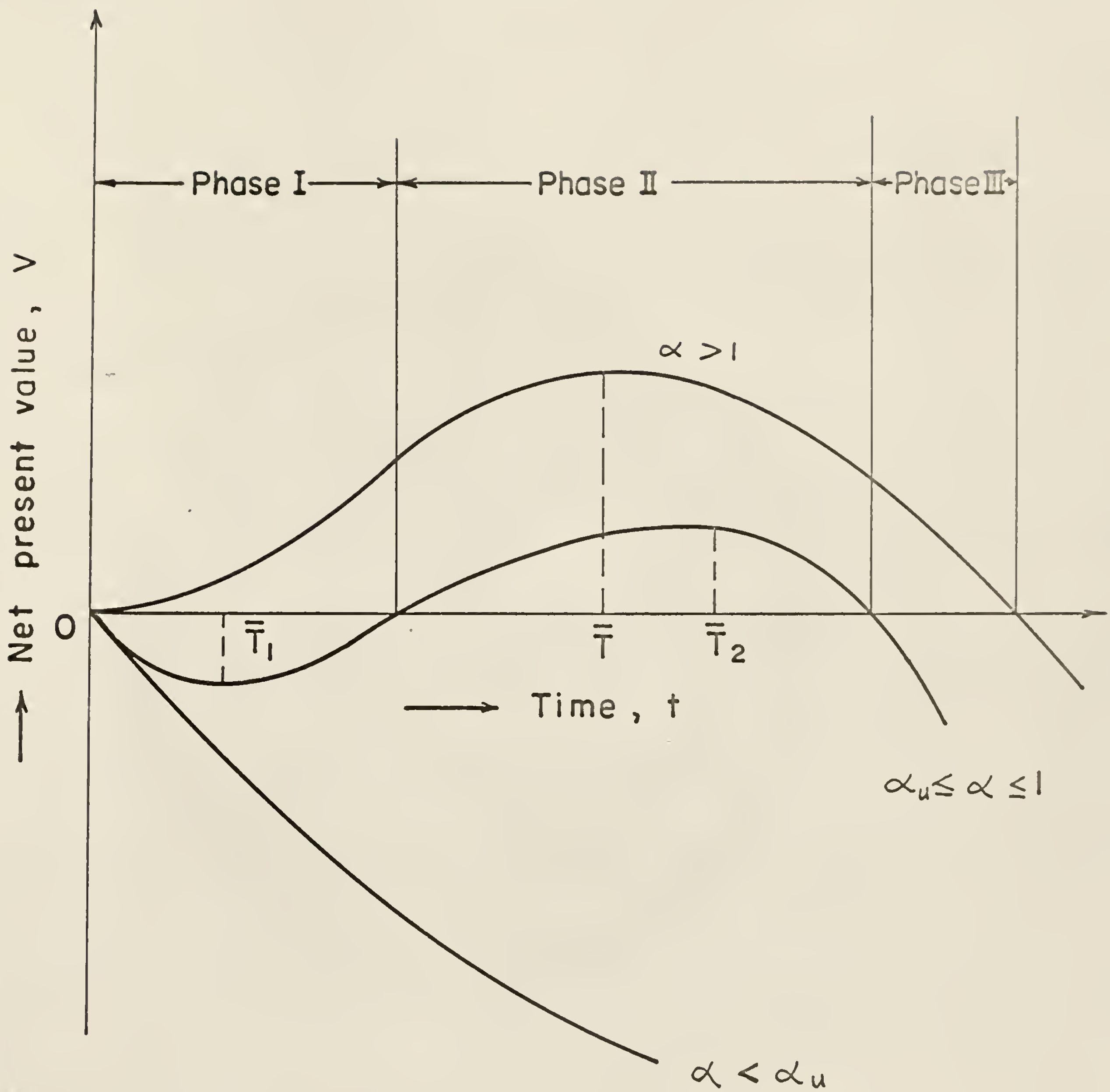

Fig. 4 Net present value under three conditions for α.

overhaul becomes imperative. The company's experience with similar machines in the past, however, has proven this overhaul to be economically inadvisable and the practice has been to dispose of the equipment by salvaging it when or before this major overhaul becomes necessary. The equipment depreciates exponentially with an annual rate of 0.30.

Although the life of the equipment is not directly affected by the manufacturing process employed, variable costs and maintenance costs differ from method to method. For all three methods, however, maintenance costs are expected to approximate the function given by equation (30). From past data, the values of the parameter, m, as well as the variable costs associated with each manufacturing method have been estimated as given in Table 1. With the existing plant facilities, the maximum rate of production $P_M(t)$, that can be achieved is 3500 units a year.

On the basis of this information, four questions are to be answered:

1. Should the company invest?
2. Which manufacturing method should be used?
3. What should the production rate be?
4. When should the company salvage the equipment?

It is the company's practice to use a rate of interest of 10% per year and the net present worth criteria in evaluating all investments.

In order to answer these questions, each one of the alternatives should be fully evaluated. Equation (68) gives the time at which the maximum present worth and/or maximum losses occur. Using these results, the maximum or minimum present worth for each

alternative can be calculated from equation (82).

A complete summary of the results is given in Table 2. It can be seen that method I generates the maximum present worth of \$2602.66 after the optimum investment period of 3.03 years. Furthermore, method I having $\alpha > 1$, gives rise to a single extremum in the net present worth function while method II, with $\alpha_u \leq \alpha \leq 1$, displayed two extrema: the first at the time where the maximum loss occurs and the second at the point of the maximum profit. Method III, for which $\alpha < \alpha_u$, gives rise to no extrema points. These results agree with our mathematical analysis.

Figure 5 is a graphical representation of the effect of decreasing the production rate below the market rate, M_s, (the optimal production rate) for the case of production method II. Extensive numerical simulation of the three production methods also confirmed that the optimal policies and the resulting minimum described values for the net present worth are indeed correct.

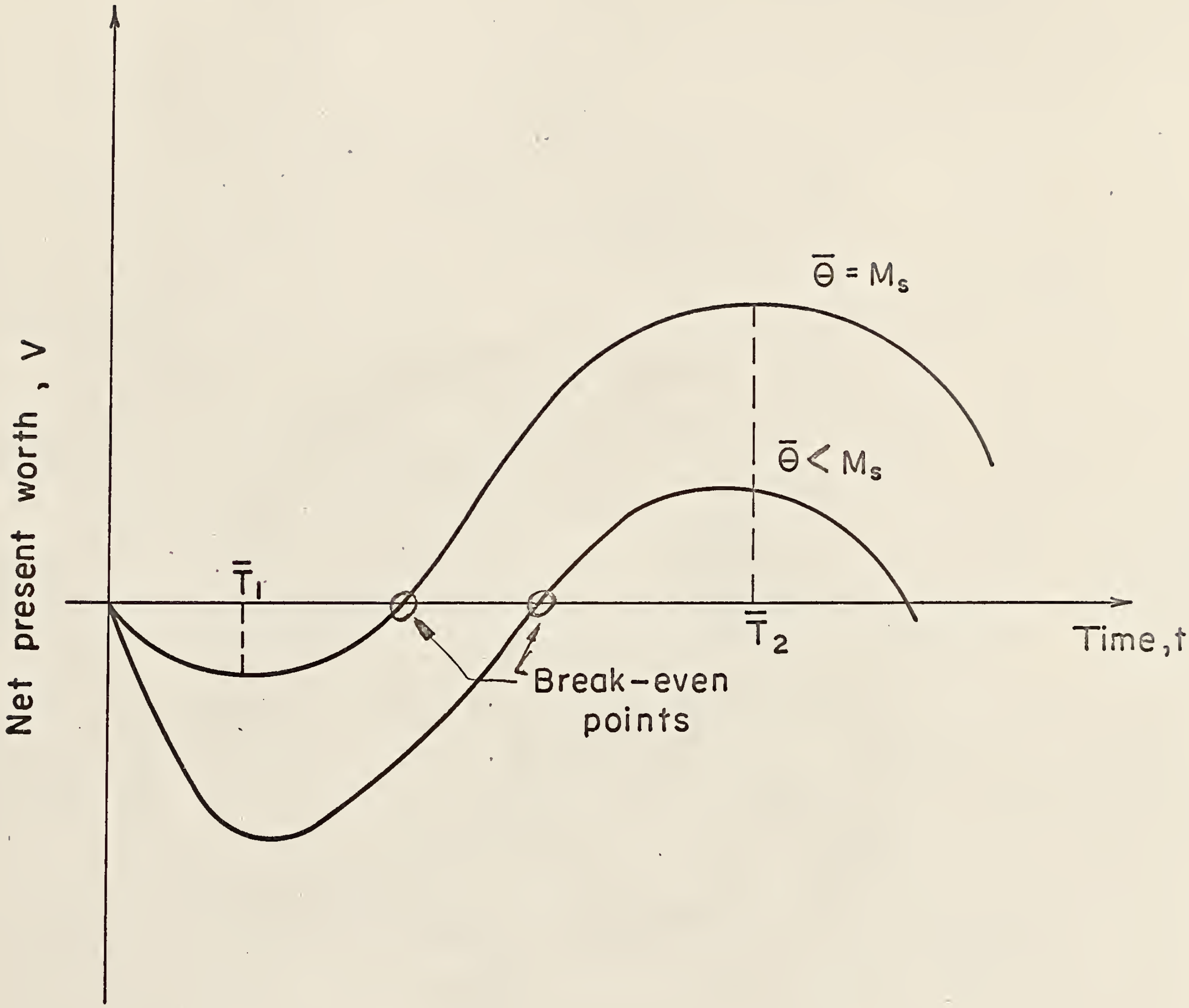

Fig. 5 Effect of decreasing the production level below the market rate (the optimum rate) (Method II).

Table 1. PARAMETRIC VALUES

Market Share, M_s = 3,000 units/year

Plant Capacity, $P_M(t)$ = 3,500 units/year

Expected Life, A = 10,000 units

Equipment Cost, B = \$10,000

Overhead Cost, E = \$2,000 per year

Depreciation Rate, k = 0.30 (exponential)

Interest Rate, i = 10% per year

Sale Price, S_p = \$5.00 per unit

	Method I	Method II	Method III
Variable Cost per unit, C_v	\$2.80	\$3.25	\$4.32
Parameter, m	1.80	0.45	0.33

Table 2. RESULTS

Method	α	α_u	β	$\overline{T}$ (Years)	Present worth, V ($)
I	1.150	0.840	0.081	3.03	2602.66
II	0.813	0.783	0.020	0.74	- 254.18
				5.54	1900.31
III	0.010	0.592	0.015	--	--

The optimum production rate for all three methods

= The market share = 3000 units/year.

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7. CONCLUDING REMARKS

The assumptions on which mathematical models are built are largely responsible for the validity of the results and the usefulness of the models. Under particular situations it might be felt that some of the assumptions undertaken by the models treated in this paper are inadequate and that, in order to reflect actual conditions, these models must take simpler forms or that additional functional characteristics should be considered.

Let us mention, however, that it was not our primary intention to develop these models but rather to demonstrate that the application of the maximum principle in optimizing these models is feasible and practical. This, we feel, has been achieved.

Furthermore, the application of the maximum principle to similar management systems needs to differ from ours only in the handling of more diversified functions. Let us not conclude, however, that the treatment by the maximum principle of the problems presented in this paper has been an exhaustive one. The maximum principle in the area of industrial management is a fairly new technique still in its developmental stages and further refinements and improvements in its theory will provide in the future for a more powerful analysis in a wider range of applications. These are, at least, our expectations.

8. ACKNOWLEDGEMENTS

The author wishes to express his appreciation to his major professor, Dr. C. L. Hwang for his guidance, constructive criticism and personal interest in the work of his student.

APPENDIX I

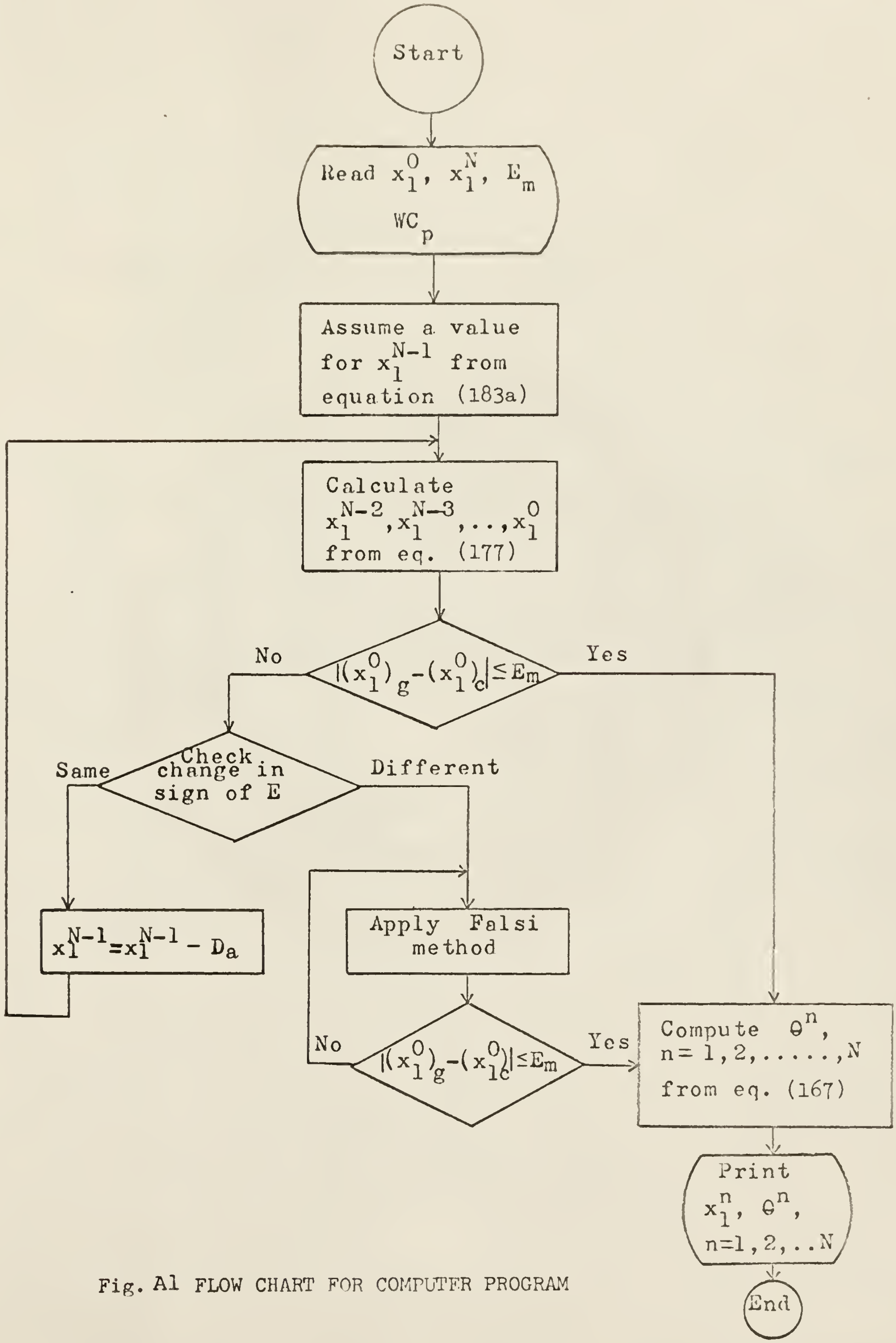


Fig. A1 FLOW CHART FOR COMPUTER PROGRAM

TABLE A1 COMPUTER PROGRAM FOR OPTIMIZATION

OF A SIMPLE HEAT EXCHANGER TRAIN

```
C  C   COMPUTATION BASED ON ESTIMATE OF X(N-1).-  DACCARETT
      DIMENSION X(10),AREA(10),U(10),CU(10),T(10),E(3),AA(3)
  101 READ 500,NS,WCP,EM
      READ 600,X0,X(NS+1)
      DO  5 K=1,NS
      J=K+1
    5 READ 600,U(J),T(J)
      PUNCH 1000
      I=1
      KS=NS-1
      CU(NS+1)=U(NS+1)/WCP
      ANS=NS
      D=(X(NS+1)-X0)/(4.5*ANS)
      X(NS)=U(NS+1)*(X(NS+1)-T(NS+1))/U(NS)+T(NS)
   10 DO 20 K=1,KS
      N=NS+1-K
      CU(N)=U(N)/WCP
      F1=X(N)-T(N)
      IF(ABS(F1)-1.0) 15,13,13
   13 F2=X(N+1)-T(N+1)
      IF(ABS(F2)-1.0) 15,18,18
   15 X(NS)=X(NS)-D
      GO TO 10
   18 X(N-1)=X(N)+F1*(CU(N)*F1/(CU(N+1)*F2)-1.0)
   20 CONTINUE
      TA=0.0
      DO 30 J=1,NS
      N=J+1
      AREA(N)=(X(N)-X(N-1))/((T(N)-X(N))*CU(N))
      TA=TA+AREA(N)
   30 CONTINUE
      E(I)=X0-X(1)
      AA(I)=X(NS)
      PUNCH 700,X(1)
      PUNCH 800,NS,X(NS+1)
      DO 33 K=1,NS
      J=K+1
   33 PUNCH 900,K,X(J),AREA(J)
      PUNCH 200,TA
      IF(ABS(E(I))-EM) 100,100,35
   35 IF(I-2) 40,45,45
   40 X(NS)=X(NS)-D
      I=2
      GO TO 10
   45 IF(E(1)*E(I)) 60,50,50
   50 E(1)=E(I)
      AA(1)=AA(I)
      IF(I-3) 55,65,65
   55 X(NS)=X(NS)-D
      GO TO 10
   60 E(2)=E(I)
```

TABLE A1 (Cont'd)

```
      AA(2)=AA(I)
   65 G=(AA(1)*ABS(E(2))+AA(2)*ABS(E(1)))/(ABS(E(1))+ABS(E(2)))
      X(NS)=G
      I=3
      GO TO 10
  100 GO TO 101
  200 FORMAT(9X 11HTOTAL AREA=,F15.3)
  500 FORMAT( I2,2F10.2)
  600 FORMAT(2F10.2)
  700 FORMAT(/10X 24HOPTIMAL DESIGN FOR X 0 =,F15.3)
  800 FORMAT(25X 5HAND X,I2,1X 1H=,F15.3)
  900 FORMAT(I5,2F15.3)
 1000 FORMAT(5HSTAGE,5X 10HEXIT TEMP.,5X 10HSTAGE AREA)
      END
```

DATA CARDS

```
030100000.000000000.01        NS,WCP,EM
0000100.000000500.00          X0,X(N+1)
0000120.000000300.00          U(1),T(1)
0000080.000000400.00          U(2),T(2)
0000040.000000600.00          U(3),T(3)
```

Program symbol	Mathematical symbol	Item
NS	N	Number of stages
N	n	Stage number
XO	$(x_1^0)_{given}$	Given value for the inlet temperature
X(N)	x_1^n	Outlet temperature of the cold stream at the n-th stage
U(N)	u^n	
CU(N)	U^n	
T(N)	t_1^n	Inlet temperature of the hot stream at the n-th stage
AREA(N)	θ_1^n	
E(I)=XO - X(1)	E	Error function
D	D_a	Decrements in x_1^{N-1} for each trial
E(1)		Value of the error before a change in sign
AA(1)	A	
E(2)		Value of the error after a change in sign
AA(2)	B	
AA(3)	G	
EE(3)		Value of the error at $x_1^{N-1} = G$
WCP	$(W)(C_P)$	

APPENDIX II

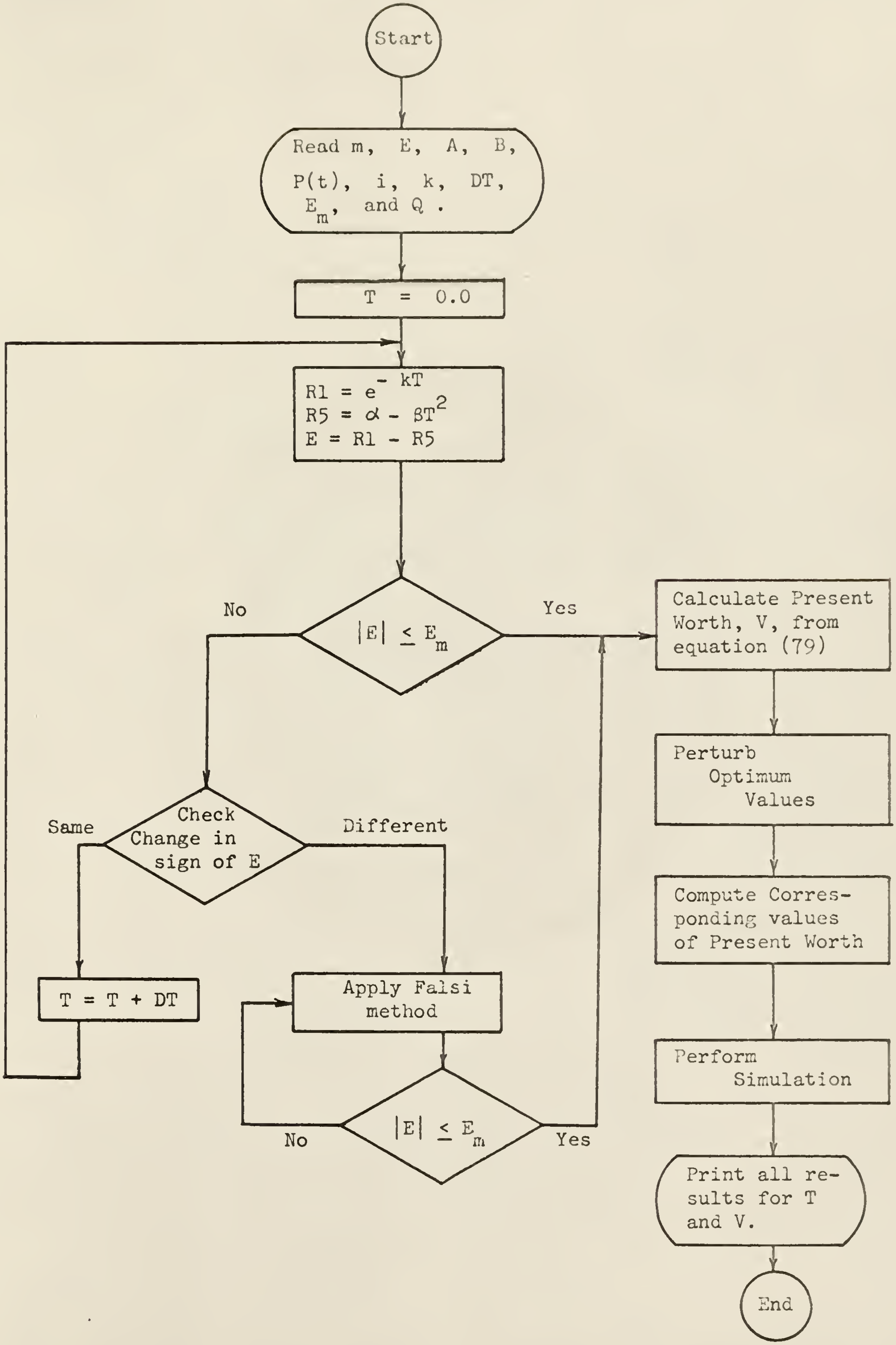
TABLE 4 COMPUTER PROGRAM

```
C   C   OPTIMIZATION OF EQUIPMENT INVESTMENT -- [illegible]
        DIMENSIONQ(5),AM(5),E(5),AA(5),ALFA(5),BETA(5),COST(5)
        READ 45 ,EE,AI,AK,B,THETA,A
        READ 46 , DT,EM
        PUNCH 455,A
        PUNCH 454,B
        PUNCH 453,AK
        PUNCH 452,AI
        PUNCH 451,EE
        PUNCH 457,THETA
        PUNCH 456,DT,EM
C
C       COMPUTING OPTIMUM INVESTMENT TIME
C
        DO 120 J=1,3
        READ 500,Q(J),AM(J)
        R2=Q(J)*THETA-EE
        R3=(AK+AI)*B
        R4=AM(J)*EE*THETA*THETA/(A*A)
        ALFA(J)=R2/R3
        BETA(J)=R4/R3
        PUNCH 461,Q(J)
        PUNCH 462,AM(J)
        PUNCH 463,ALFA(J),BETA(J)
        NR=0
        T=0.0
      5 I=1
     10 R1=1.0/EXP(AK*T)
        R5=(R2-R4*T*T)/R3
        E(I)=R1-R5
        AA(I)=T
        IF(ABS(E(I))-EM) 100,100,35
     35 IF(I-2) 40,45,45
     40 T=T+DT
        I=2
        GO TO 10
     45 IF(E(1)*E(2)) 60,50,50
     50 E(1)=E(I)
        AA(1)=AA(I)
        IF(I-3) 55,65,65
     55 IF(R5) 120,120,56
     56 T=T+DT
        GO TO 10
     60 E(2)=E(I)
        AA(2)=AA(I)
     65 G=(AA(1)*ABS(E(2))+AA(2)*ABS(E(1)))/(ABS(E(1))+ABS(E(2)))
        T=G
        I=3
        GO TO 10
C
```

```
C       ENTER SIMULATION PROCESS
C
        PUNCH 550
        PUNCH 555
        PUNCH 56
        PUNCH 565,ALFA(1),ALFA(2),ALFA(3)
        PUNCH 57 ,BETA(1),BETA(2),BETA(3)
        PUNCH 575
        T=0.
        DO 140 LL=1,50
        DO 130 J=1,3
        R1=1. /EXP(AK*T)
        R2=Q(J)*THETA-EE
        R4=AM(J)*EE*THETA*THETA/(A*A)
        F1=R2/AI
        F2=1. /EXP(AI*T)
        F3=(1.0-F2)*F1
        F4=B*(R1*F2-1.0)
        F5=R4/(AI*AI*AI)
        F6=AI*AI*T*T+2.0*AI*T+2.0
        VALUE=F3+F4+F5*(F2*F6-2.0)
  130   WORTH(J)=VALUE
        PUNCH 580,T,WORTH(1),WORTH(2),WORTH(3)
  140   T=T+0.25
C
  103 FORMAT (/15X 38HPERTURBATION OF OPTIMAL VALUES FOLLOW)
  157 FORMAT(/10X 18HNUMBER OF EXTREMA=,I2)
  200 FORMAT(/10X 31HOPTIMUM VALUES FOLLOW (ROOT NO.,I2,1H))
  300 FORMAT(/10X 5HTIME=,F6.2,5X 10HNET WORTH=,F10.2)
  400 FORMAT(15X 5HTIME=,F6.2,3X 5HRATE=,F8.2,3X 6HWORTH=,F10.2)
  450 FORMAT(6F10.2)
  451 FORMAT(23HFIXED OVERHEAD COST= $ ,F10.2)
  452 FORMAT(14HINTEREST RATE=,F5.2)
  453 FORMAT(18HDEPRECIATION RATE=,F5.2)
  454 FORMAT(25HNET INSTALLATION COST= $ ,F10.2)
  455 FORMAT(25HEXPECTED SERVICE (UNITS)=F10.2)
  456 FORMAT( 3HDT=,F5.2,5X 11HMAX. ERROR=,F8.5)
  457 FORMAT(21HMAXIMUM MARKET SHARE=,F10.2)
  460 FORMAT(2F10.5)
  461 FORMAT(//24HUNIT LOGISTIC MARGIN= $ ,F5.2)
  462 FORMAT(/4HM = ,F5.2)
  463 FORMAT(/5HALFA=,F6.3,5X 5HBETA=,F6.3)
  50  FORMAT(2F10.3)
  550 FORMAT(//20X20H*****SIMULATION*****)
  555 FORMAT(//8X 44HNET PRESENT WORTH UNDER THREE CONDITIONS ($))
  560 FORMAT(/6X 9HTIME(YRS),8X 7HCOND. 1,8X 7HCOND. 2,8X 7HCOND. 3)
  565 FORMAT(/19X5HALFA=,F6.3,4X 5HALFA=,F6.3,4X 5HALFA=,F6.3)
  570 FORMAT(19X5HBETA=,F6.3,4X 5HBETA=,F6.3,4X 5HBETA=,F6.3)
  575 FORMAT(1H )
  580 FORMAT(4F15.2)
        END
C       DATA CARDS
0002000.000000 .1 [illegible] .3 [illegible] 1 [illegible] . [illegible] . [illegible] 1 [illegible] .
0000.400000000.01[illegible]
000002.200000 1.8
000001.75000000 .445
000000.6800000000.333
```

```
C       COMPUTING MAXIMUM PRESENT WORTH
C
  100 NR=NR+1
      PUNCH 290,NR
      F1=R2/AI
      F2=1.0/EXP(AI*T)
      F3=(1.0-F2)*F1
      F4=B*(R1*F2-1.0)
      F5=R4/(AI*AI*AI)
      F6=AI*AI*T*T+2.0*AI*T+2.0
      VALUE=F3+F4+F5*(F2*F6-2.0)
      PUNCH 300,T,VALUE
C
C       PERTURBING OPTIMUM VALUES
C
      PUNCH 103
      BEST=T
      EYE=0.0
      DO 113 L1=1,2
      IF(EYE) 1000,1000,2000
 1000 PRO=THETA
      EYE=1.0
      GO TO 105
 2000 PRO=0.95*THETA
  105 OJO=0.0
      DO 113 L2=1,2
      IF(OJO) 1500,1500,2500
 1500 T=0.95*BEST
      OJO=1.0
      GO TO 110
 2500 T=1.05*BEST
  110 RR1=1.0/EXP(AK*T)
      RR2=Q(J)*PRO-EE
      RR4=AM(J)*EE*THETA*THETA/(A*A)
      FF1=RR2/AI
      FF2=1.0/EXP(AI*T)
      FF3=(1.0-FF2)*FF1
      FF4=B*(RR1*FF2-1.0)
      FF5=RR4/(AI*AI*AI)
      FF6=AI*AI*T*T+2.0*AI*T+2.0
      VALUE=FF3+FF4+FF5*(FF2*FF6-2.0)
  113 PUNCH 400,T,PRO,VALUE
      IF(R5) 120,120,115
  115 T=BEST+DT
      GO TO 5
  120 PUNCH 157,NR
C
```

Fig. 6 FLOW CHART



Program symbol	Mathematical symbol	Item
EE	E	Fixed overhead cost
AI	i	Interest rate
AK	k	Depreciation rate
B	B	Total installation cost
THETA	P(t)	Production rate
A	A	Expected service
A1FA(J)	α	Parameter
BETA(J)	β	Parameter
AM	m	Parameter
T	$\bar{T}$	Optimum time
(Computing Investment Time)		
E(1)		Value of error before a change in sign
AA(1)		Value of $\bar{T}$ at this point
E(2)		Value of error after a change in sign
AA(2)		Value of $\bar{T}$ at this point
AA(3) G		Interpolation between AA(1) and AA(2)
E(3)		Value of error at AA(3)
(Computing Optimum Present Worth)		
NR		Number of roots in eq. (68) and Number of extrema points in eq. (79)
VALUE	$\bar{V}$	Net Present Worth

Program symbol	Mathematical symbol	Item
(Perturbing Optimum Values)		
PRO		Production Rate
BEST	$\bar{T}_1$, $\bar{T}_2$	Optimum times
EYE, OJO		Dummy variables
(Simulation Processes)		
WORTH(J)	V	Net present value

SYMBOL TABLE

TABLE 5 RESULTS

```
C  C  OPTIMIZATION OF EQUIPMENT INVESTMENT -- [illegible]

EXPECTED SERVICE (UNITS)=  1000.0
NET INSTALLATION CONST= $  1000.0
DEPRECIATION RATE=  .30
INTEREST RATE=  .100
FIXED OVERHEAD COST= $    2000.00
MAXIMUM MARKET SHARE=   3000.00
DT=  .40      MAX. ERROR=  .01000


UNIT LOGISTIC MARGIN= $  2.20

M =  1.80

ALFA= 1.150      BETA=  .081

          OPTIMUM VALUES FOLLOW (ROOT NO. 1)

          TIME=  3.03       NET WORTH=   2602.66

               PERTURBATION OF OPTIMUM VALUES FOLLOWS
               TIME=  2.88    RATE= 3000.00    WORTH=   25[illegible].59
               TIME=  3.18    RATE= 3000.00    WORTH=   2591.61
               TIME=  2.88    RATE= 2850.00    WORTH=   1763.47
               TIME=  3.18    RATE= 2850.00    WORTH=   1692.65

          NUMBER OF EXTREMA= 1


UNIT LOGISTIC MARGIN= $  1.75

M =   .45

ALFA=  .813      BETA=  .020

          OPTIMUM VALUES FOLLOW (ROOT NO. 1)

          TIME=   .74       NET WORTH=   -254.18

               PERTURBATION OF OPTIMUM VALUES FOLLOWS
               TIME=   .71    RATE= 3000.00    WORTH=   -253.79
               TIME=   .78    RATE= 3000.00    WORTH=   -253.4[illegible]
               TIME=   .71    RATE= 2850.00    WORTH=   -432.[illegible]5
               TIME=   .78    RATE= 2850.00    WORTH=   -4[illegible].35

          OPTIMUM VALUES FOLLOW (ROOT NO. 2)

          TIME=  5.54       NET WORTH=   19[illegible].31

               PERTURBATION OF OPTIMUM VALUES FOLLOWS
               TIME=  5.27    RATE= 3000.00    WORTH=   1880.[illegible]
               TIME=  5.82    RATE= 3000.00    WORTH=   18[illegible].42
               TIME=  5.27    RATE= 2850.00    WORTH=    806.41
               TIME=  5.82    RATE= 2850.00    WORTH=    732.29

          NUMBER OF EXTREMA= 2
```

UNIT LOGISTIC MARGIN= [illegible] .0[illegible]

M = .33

ALFA= .[illegible]1[illegible] BETA= . 15

NUMBER OF EXTREMA= 0

*****SIMULATION*****

NET PRESENT WORTH UNDER THREE CONDITIONS ([illegible])

TIME(YRS)	COND. 1	COND. 2	COND. 3
	ALFA= 1.150 BETA= .081	ALFA= .01[illegible] BETA= .[illegible]2[illegible]	ALFA= .[illegible]1[illegible] BETA= .[illegible]
0.00	.00	[illegible]	[illegible].00
.25	182.47	-14[illegible].61	-[illegible]42.[illegible]6
.50	417.79	-23[illegible].85	-1705.[illegible]
.75	688.92	-254.13 ←	-2570.[illegible]
1.00	98[illegible].47	-228.79	-3277.27
1.25	1278.35	-16[illegible].[illegible]	-3[illegible]23.23
1.50	1569.73	-65.44	-4516.44
1.75	1842.93	57.[illegible]	-5063.[illegible]
2.00	2[illegible]87.48	20[illegible].[illegible]6	-5571.[illegible]
2.25	2293.85	356.76	-6046.1[illegible]
2.50	2453.31	521.[illegible]	-64[illegible].[illegible]
2.75	2558.34	69[illegible].1[illegible]	-691[illegible].[illegible]
3.00	2601.89 ←	856.73	-7315.6[illegible]
3.25	2577.76	1[illegible]23.44	-77[illegible]2.2[illegible]
3.50	248[illegible].55	1181.[illegible]	-[illegible].21
3.75	2[illegible]5.37	1326.72	-[illegible]
4.00	2[illegible]48.[illegible]4	1463.78	-[illegible]7[illegible]
4.25	1704.73	15[illegible]	-[illegible]
4.50	1272.34	1607.72	-[illegible].27
4.75	748.09	1771.[illegible]2	-[illegible]
5.00	129.75	1036.21	-1[illegible]214.[illegible]
5.25	-584.57	1075.[illegible]6	-1[illegible]72.[illegible]1
5.50	-1396.40	1099.21 ←	-1[illegible]936.2[illegible]
5.75	-23[illegible]6.87	1855.71	-113[illegible]6.[illegible]
6.00	-3316.84	1567.74	-11[illegible]
6.25	-4426.8[illegible]	1[illegible]	-12[illegible]
6.50	-5637.18	17[illegible].[illegible]	-12463.76
6.75	-6947.79	163[illegible].92	-12[illegible]
7.00	-8358.55	14[illegible].[illegible]	-132[illegible].42
7.25	-9869.[illegible]	1341.67	-1[illegible]7[illegible].4[illegible]
7.50	-11476.4[illegible]	115[illegible]	-14[illegible]
7.75	-13180.26	9[illegible].54	-14[illegible]6.[illegible]
8.00	-14991.32	[illegible]7.[illegible]4	-15[illegible]57.[illegible]
8.25	-16892.57	[illegible]	-15[illegible]
8.50	-1888[illegible].72	15[illegible]	-16[illegible]
8.75	-2[illegible]978.48	-16[illegible]	-1[illegible]

9.00	-2316[illegible].24	-5[illegible]	-17[illegible]24.6[illegible]
9.25	-25432.47	-[illegible]7[illegible].1[illegible]	-17547.[illegible]
9.50	-27793.44	-127[illegible].7[illegible]	-18[illegible].[illegible]
9.75	-30241.43	-[illegible]69[illegible].[illegible]	-18630.[illegible]
10.00	-32774.57	-2137.2	-1[illegible]1[illegible].52
10.25	-35390.98	-2605.25	-19762.43
10.50	-38088.59	-3[illegible]70.[illegible]7	-20346.18
10.75	-40865.46	-3611.[illegible]7	-20941.6[illegible]
11.00	-43719.53	-414[illegible].84	-2154[illegible].47
11.25	-46648.66	-47[illegible].14	-22166.[illegible]
11.50	-4965[illegible].71	-5288.[illegible]	-22795.[illegible]
11.75	-52723.56	-5890.63	-23435.21
12.00	-55865.00	-6512.68	-24085.27
12.25	-59072.78	-7154.50	-24745.40

0 END OF PROGRAM AT STATEMENT 0580 + [illegible] LINES

OPTIMUM INDUSTRIAL MANAGEMENT SYSTEMS BY THE MAXIMUM PRINCIPLE

By

JOSE ENRIQUE DACCARETT

B. S., Wichita State University, 1965

AN ABSTRACT OF A MASTER'S THESIS

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MASTER OF SCIENCE

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A remarkable growth of interest in problems of dynamic optimization has given rise during the past decade to a number of methods useful for rendering systems optimal. One such method is Pontryagin's maximum principle.

Originally formulated in 1956 by the Russian mathematician, the maximum principle was intended for the optimization of continuous control systems. In 1959, the first attempt to extend the maximum principle to the optimization of stagewise processes was made by Rozonoer. Several subsequent versions of the maximum principle were then advanced.

The application of the maximum principle to management and operations research is still very limited. The objective of this thesis is to demonstrate the applicability of the maximum principle to some problems in the area of management and industrial engineering, concentrating mainly on problems belonging to the continuous case. The maximum principle is applied in the optimization of already developed models and functional variations of these models. Some numerical examples are presented for further clarification of the treatments.

The basic algorithms of the discrete and the continuous maximum principle are presented first, and then the discrete version is applied in order to optimize the temperatures of a multistage heat exchanger. The solution of the resulting two-point boundary value is demonstrated in detail.

A model for sales response to advertising is treated by the continuous maximum principle, and then the linear constraint on the response function is removed. The treatment of this model leads to three key advertising policies.

Next, a continuous model for production planning is studied and, finally, two models for the optimization of equipment investment based on the net present value are treated by the maximum principle. Two numerical examples supplement the treatment of these models.

The efficiency of the maximum principle in dealing with this class of problem is not compared with that of other methods. The reason for this is that the application of the maximum principle to this sort of problems has not left the incipient stages of development. This is a new technique and as such, due refinements and further developments must take place before any comparisons of computational efficiency can be made.